Exponential integrators and two-scale approaches for unsaturated water flow in heterogeneous soils

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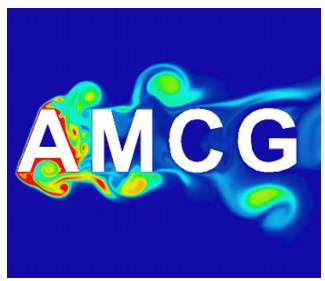

NORMS Meeting, 26 November 2014

Overview of Presentation

- Modelling of unsaturated water flow
- Time integration methods for unsaturated water flow: Exponential integrators
- Two-scale numerical modelling of unsaturated water flow in soils exhibiting small-scale heterogeneities

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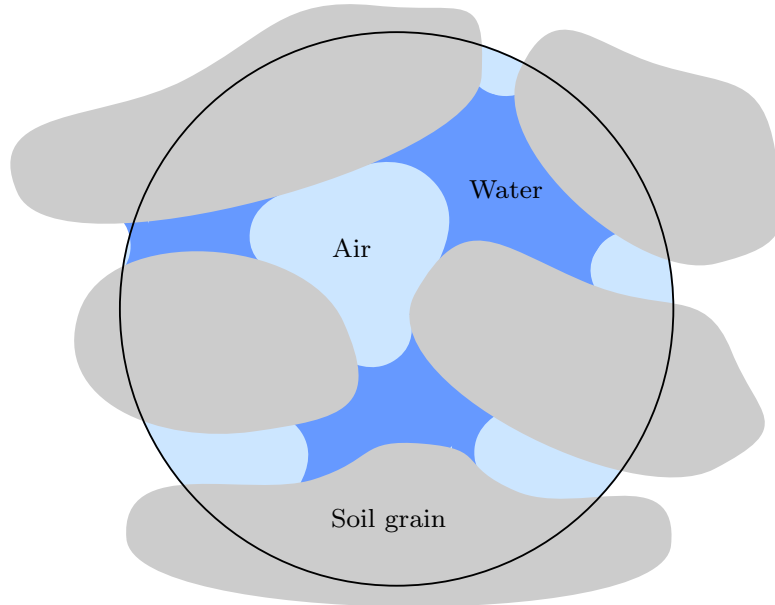
Unsaturated water flow

- Immiscible two-phase (air and water) flow
- Darcy's Law:

$$\boldsymbol{q}_\alpha = -\frac{k_{r,\alpha}}{\mu_\alpha} K \left(\nabla p_\alpha - \rho_\alpha \boldsymbol{g} \right)$$

- Capillary pressure $p_c = p_a - p_w$



- Mass conservation for each phase (assuming incompressibility)

$$\sum_{\alpha \in \{a,w\}} \boldsymbol{q}_\alpha = \boldsymbol{0}\,, \qquad \frac{\partial \left(\phi S_\alpha \right)}{\partial t} + \nabla \cdot \boldsymbol{q}_\alpha = 0$$

- Sum of saturations $S_w + S_a = 1$

Richards' equation

- Assumptions:
  (a) air phase is at constant and atmospheric pressure
  (b) gravity acts in the negative y direction (2D)

- Single equation for the water phase:

$$\frac{\partial \theta(h)}{\partial t} + \nabla \cdot (-K(h) \nabla (h + x_2)) = 0$$

  where $h = p_c/(\rho_w g)$ is the capillary pressure head, $\theta = \phi S_w$ is the moisture content and K is the hydraulic conducitivity.

- Closure relationships [van Genuchten (1980)]:

$$\theta(h) = \theta_r + (\theta_s - \theta_r) S_e(h)$$

$$K(h) = K_{\text{sat}} \sqrt{S_e} \left(1 - \left(1 - S_e^{1/m}\right)^m\right)^2$$

$$S_e(h) = (1 + (-\alpha h)^n)^{-m}$$

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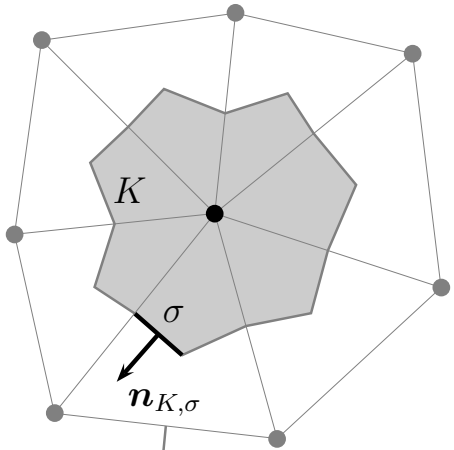
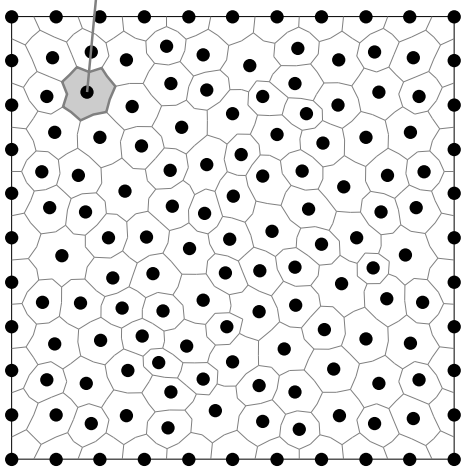
Control Volume Method



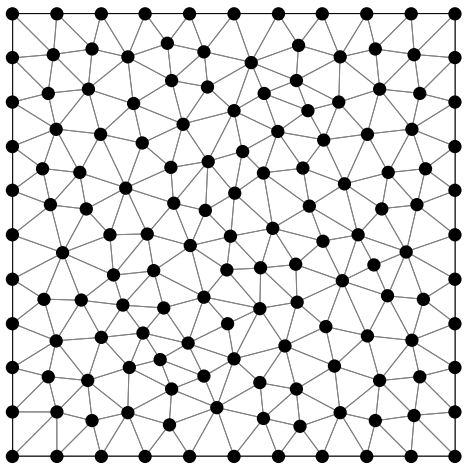
Spatial discretisation:

$$\frac{d}{dt}\int_K \theta \, dV + \sum_{\sigma \subset \partial K} \int_\sigma -K(h)\nabla\left(h + x_2\right) \cdot \boldsymbol{n}_{K,\sigma} \, dS = 0$$

CV mesh



Mesh



Time integration methods

- Spatial discretisation can be expressed in the form

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{g}(\boldsymbol{u}) \,, \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in \mathbb{R}^N \,,$$

  where $\boldsymbol{u} = (h_1, \ldots, h_N)^T$ and N is the number of nodes.

- Classical explicit and implicit methods

$$\boldsymbol{u}_{n+1} - \boldsymbol{u}_n = \int_{t_n}^{t_{n+1}} \boldsymbol{g}(\boldsymbol{u}) \, dt$$

$$\begin{aligned}
\boldsymbol{u}_{n+1} - \boldsymbol{u}_n &= \tau_n g(\boldsymbol{u}_n) && \textit{Forward Euler method} \\
\boldsymbol{u}_{n+1} - \boldsymbol{u}_n &= \tau_n g(\boldsymbol{u}_{n+1}) && \textit{Backward Euler method} \\
\boldsymbol{u}_{n+1} - \boldsymbol{u}_n &= \tau_n \left[(1-\theta)\boldsymbol{g}(\boldsymbol{u}_n) + \theta \boldsymbol{g}(\boldsymbol{u}_{n+1}) \right] && \textit{Theta method}
\end{aligned}$$

  where $\theta = 1/2$ gives the classical Crank-Nicolson method.

- Fluidity/IC-FERST uses the Theta method.

Exponential integrators

An *exponential integrator* is any time integration method that involves the exponential of the Jacobian matrix [Minchev and Wright (2005)]

Prototype method (Exponential Euler): At each time step, solve the linearised problem:

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{g}(\boldsymbol{u}) \quad \xrightarrow{\text{Linearise}} \quad \frac{d\boldsymbol{u}}{dt} = \boldsymbol{g}_n + \boldsymbol{J}_n(\boldsymbol{u} - \boldsymbol{u}_n)$$

where $\boldsymbol{g}_n = \boldsymbol{g}(\boldsymbol{u}_n)$ and $\boldsymbol{J}_n = \boldsymbol{J}(\boldsymbol{u}_n)$, exactly over a single time step:

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + \tau_n \varphi(\tau_n \boldsymbol{J}_n)\boldsymbol{g}_n$$

where $\varphi(\boldsymbol{A}) = \boldsymbol{A}^{-1}(e^{\boldsymbol{A}} - \boldsymbol{I})$.

- Explicit
- Second order [Hochbruck et al. (1998)]
- Exact for linear problems $\Rightarrow$ A-stable in the linear stability sense

Krylov subspace methods for matrix functions

- Extract an approximation from a small m dimensional Krylov subspace

$$\mathcal{K}_m(\boldsymbol{J}_n, \boldsymbol{g}_n) = \text{span}\left\{\boldsymbol{g}_n, \boldsymbol{J}_n\boldsymbol{g}_n, \ldots, \boldsymbol{J}_n^{m-1}\boldsymbol{g}_n\right\} \subset \mathbb{R}^N$$

- Construct an orthonormal basis for the subspace using Arnoldi's method

$$\boldsymbol{J}_n\boldsymbol{V}_m = \boldsymbol{V}_m\boldsymbol{H}_m + \beta_m\boldsymbol{v}_{m+1}\boldsymbol{e}_m^T$$

  where the columns of $\boldsymbol{V}_m = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m]$ form an orthonormal basis for $\mathcal{K}_m$.

- Arnoldi's method requires only matrix-vector products with $\boldsymbol{J}_n$

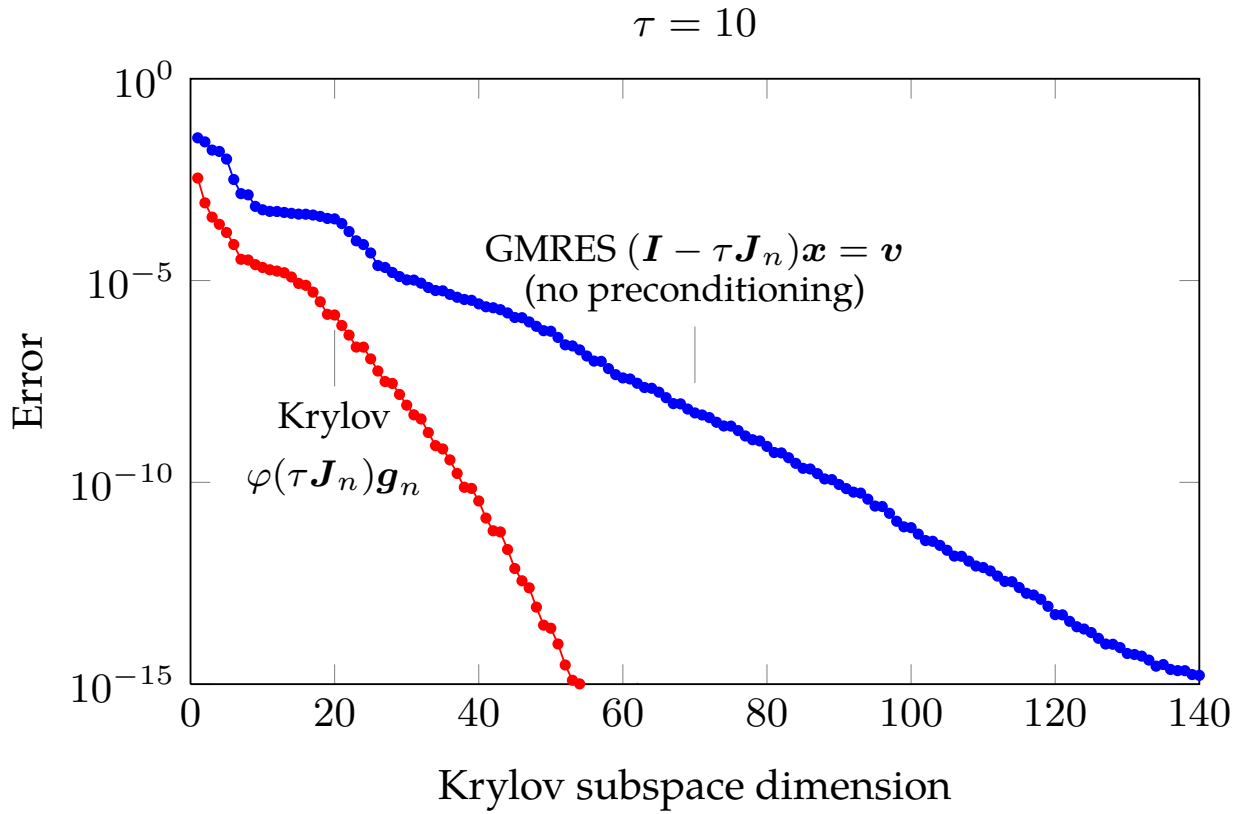
$$\boldsymbol{J}_n\boldsymbol{v} \approx [\boldsymbol{g}(\boldsymbol{u}_n + \varepsilon\boldsymbol{v}) - \boldsymbol{g}(\boldsymbol{u}_n)]/\varepsilon$$

- Standard Krylov subspace approximation [Hochbruck et al. (1998)]

$$\varphi(\tau_n\boldsymbol{J}_n)\boldsymbol{g}_n \approx \boldsymbol{V}_m\left[\|\boldsymbol{g}_n\|_2\varphi(\tau_n\boldsymbol{H}_m)\boldsymbol{e}_1\right]$$

- Very accurate approximations can be obtained for $m \ll N$ so the matrix $\varphi(\tau\boldsymbol{H}_m)$ can be computed cheaply (e.g., using Padé approximation).

One attraction of exponential integrators



Higher order exponential integrators

- Write the ODE system in the form

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{g}_n + \boldsymbol{J}_n(\boldsymbol{u} - \boldsymbol{u}_n) + \boldsymbol{r}(\boldsymbol{u}) \,,$$

  where $\boldsymbol{r}(\boldsymbol{u}) = \boldsymbol{g}(\boldsymbol{u}) - \boldsymbol{g}_n - \boldsymbol{J}_n(\boldsymbol{u} - \boldsymbol{u}_n)$.

- Using the integrating factor $e^{-t\boldsymbol{J}_n}$:

$$\frac{d}{dt}\left(e^{-t\boldsymbol{J}_n}\boldsymbol{u}(t)\right) = e^{-t\boldsymbol{J}_n}\left(\boldsymbol{g}_n - \boldsymbol{J}_n\boldsymbol{u}_n\right) + e^{-t\boldsymbol{J}_n}\boldsymbol{r}(\boldsymbol{u}) \,.$$

  the exact solution can be expressed as

$$\boldsymbol{u}(t_{n+1}) = \boldsymbol{u}_n + \tau_n\varphi_1(\tau_n\boldsymbol{J}_n)\boldsymbol{g}_n + \int_{t_n}^{t_{n+1}} e^{(t_{n+1}-t)\boldsymbol{J}_n}\boldsymbol{r}(\boldsymbol{u})\,dt \,.$$

- This formula is the starting point for developing higher order methods [Tokman (2006), Hochbruck et al. (2009)].

Backward Differentiation Formula (BDF)

- Higher-order generalisations of the Backward Euler method
- Evaluate ODE system at $t = t_{n+1}$ to give

$$\frac{d\boldsymbol{u}}{dt}(t_{n+1}) = \boldsymbol{g}(\boldsymbol{u}_{n+1})$$

- For a BDF of order q: Interpolate the solution at $(t_{n+1-i}, \boldsymbol{u}_{n+1-i})$, $i = 0, \ldots, q$ using a polynomial of degree q:

$$\boldsymbol{u}(t) \approx \sum_{i=0}^{q} \ell_i(t) \boldsymbol{u}_{n+1-i} \qquad \ell_i(t) = \prod_{\substack{j=0 \\ j \neq i}}^{q} \frac{t - t_{n+1-j}}{t_{n+1-i} - t_{n+1-j}}$$

  where $\ell_i(t)$ are the usual Lagrange interpolating polynomials.

- Differentiate interpolant and evaluate at $t = t_{n+1}$ to obtain

$$\frac{d\boldsymbol{u}}{dt}(t_{n+1}) \approx \sum_{i=0}^{q} \ell_i'(t) \boldsymbol{u}_{n+1-i}$$

Tested schemes

Exponential integrators

- Exponential Euler method (2nd order) [Carr et al. (2011)]
- Exponential Rosenbrock method (4th order) [Hochbruck et al. (2009)]
- At every time step, the error is estimated $\boldsymbol{\varepsilon}_{n+1} \approx \boldsymbol{u}_{n+1} - \hat{\boldsymbol{u}}_{n+1}$ using an embedded method $\hat{\boldsymbol{u}}_{n+1}$ and the time step adjusted to satisfy user prescribed error tolerances. Stepsize accepted if $\|\boldsymbol{\varepsilon}_{n+1}\|_{tol} \leq 1$ where

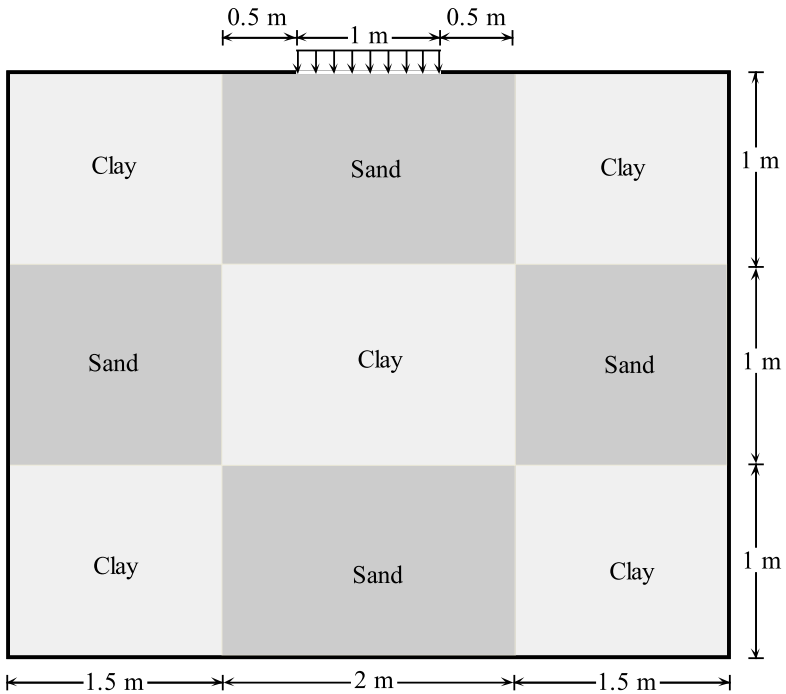
$$\|\boldsymbol{y}\|_{tol} = \|\boldsymbol{x}\|_2/\sqrt{N}\,, \qquad x_i = y_i/(tol_a + u_{n,i} tol_r)$$

Backward Differentiation Formulae (BDF)

- BDF Methods of order 2 and 4
- CVODE module of SUNDIALS [Hindmarsh et al. (2005)]
- Employs Newton iteration with an ILU preconditioned GMRES method.
- Periodically adjusts the order, with the goal of maximizing the stepsize
- Stepsize control

Test Case: Unsaturated water flow

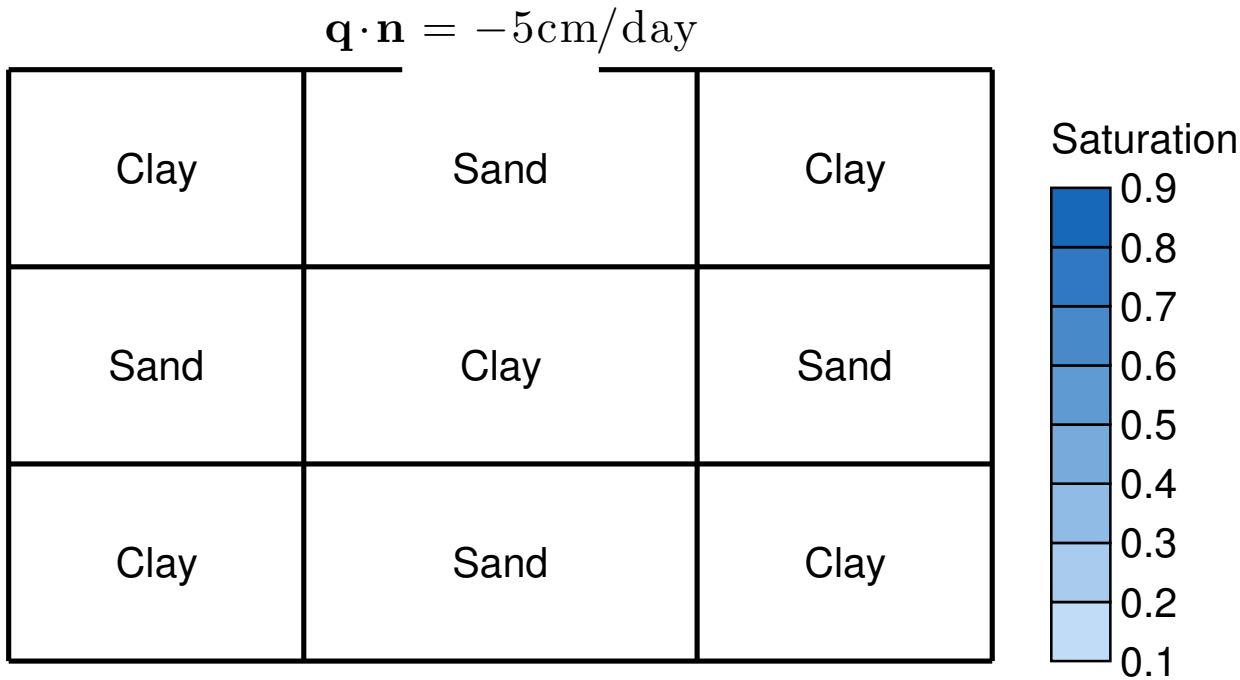
- Test Case [Kirkland et al. (1992), McBride et al. (2006), Carr et al. (2011, 2013)]



	Sand	Clay
K_{sat}	6.262e-5	1.516e-6
θ_r	0.0286	0.1060
θ_s	0.3658	0.4686
α	2.80	1.04
n	2.2390	1.3964

- Mesh: 132×132 grid, $N = 16641$.

Test Problem: Water infiltration



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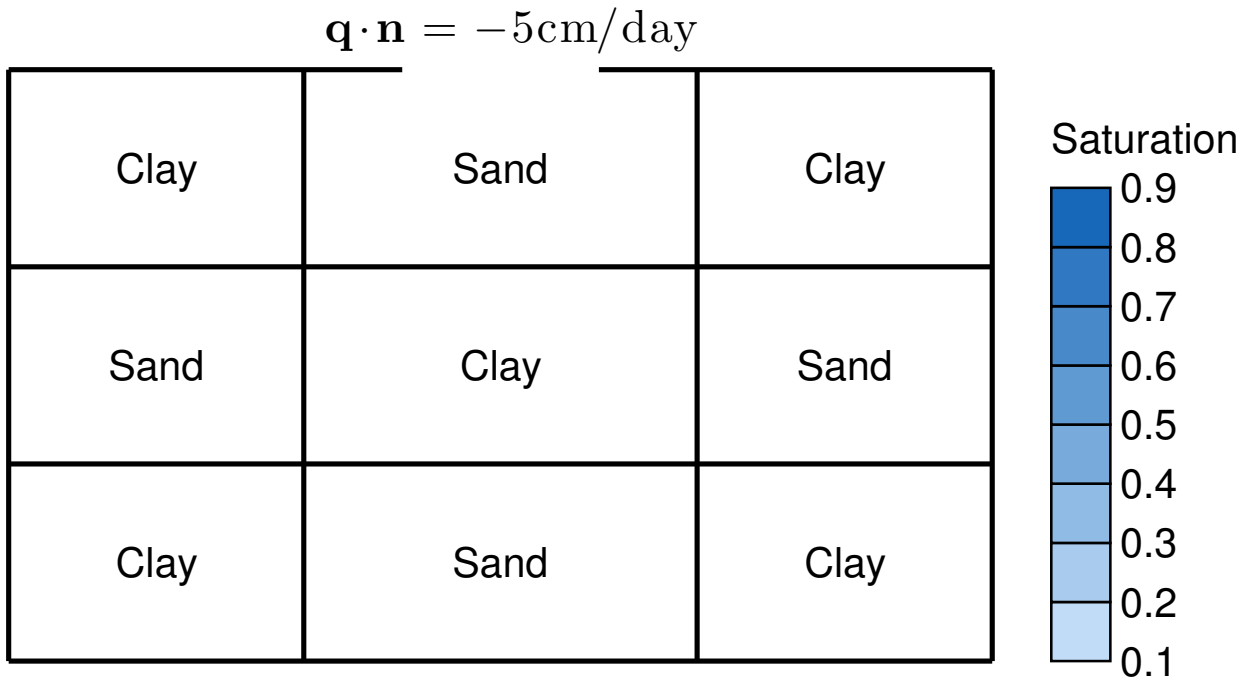

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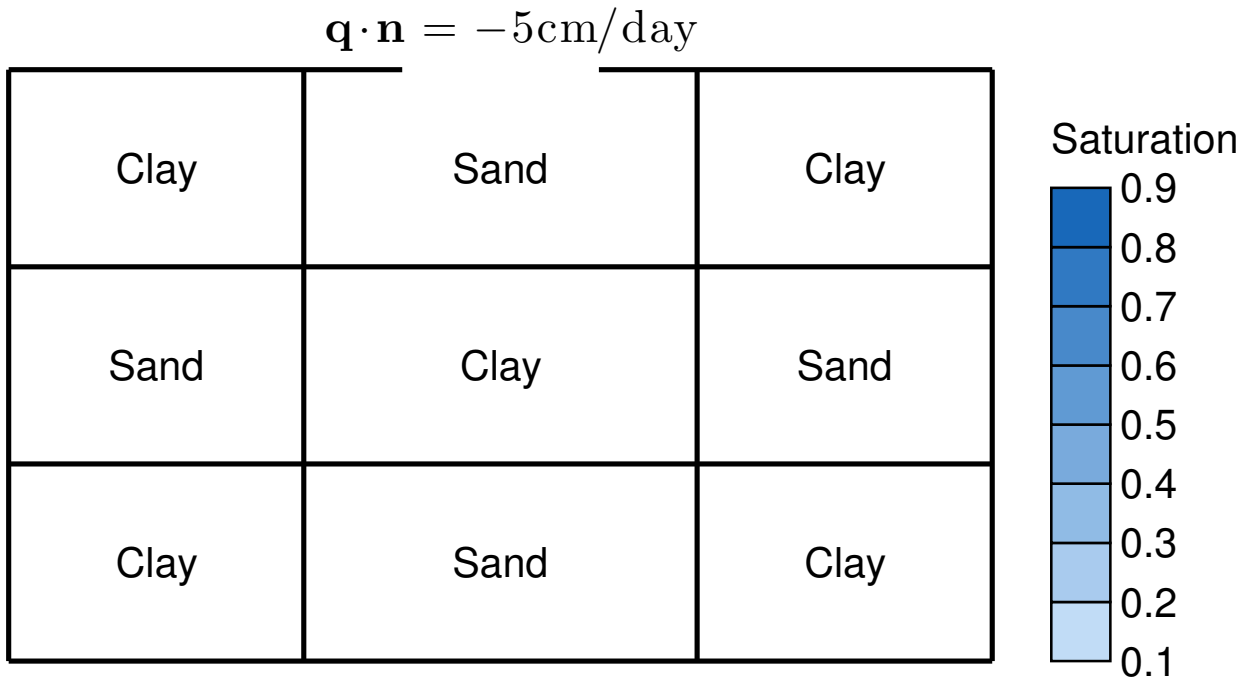

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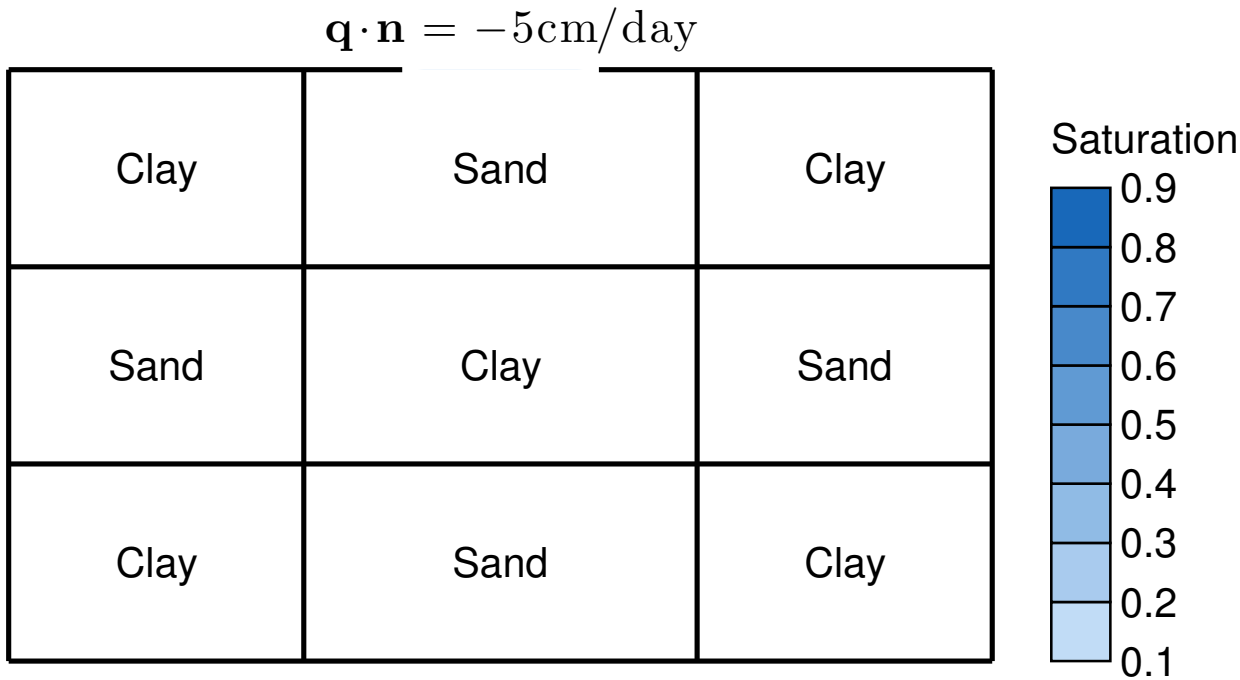

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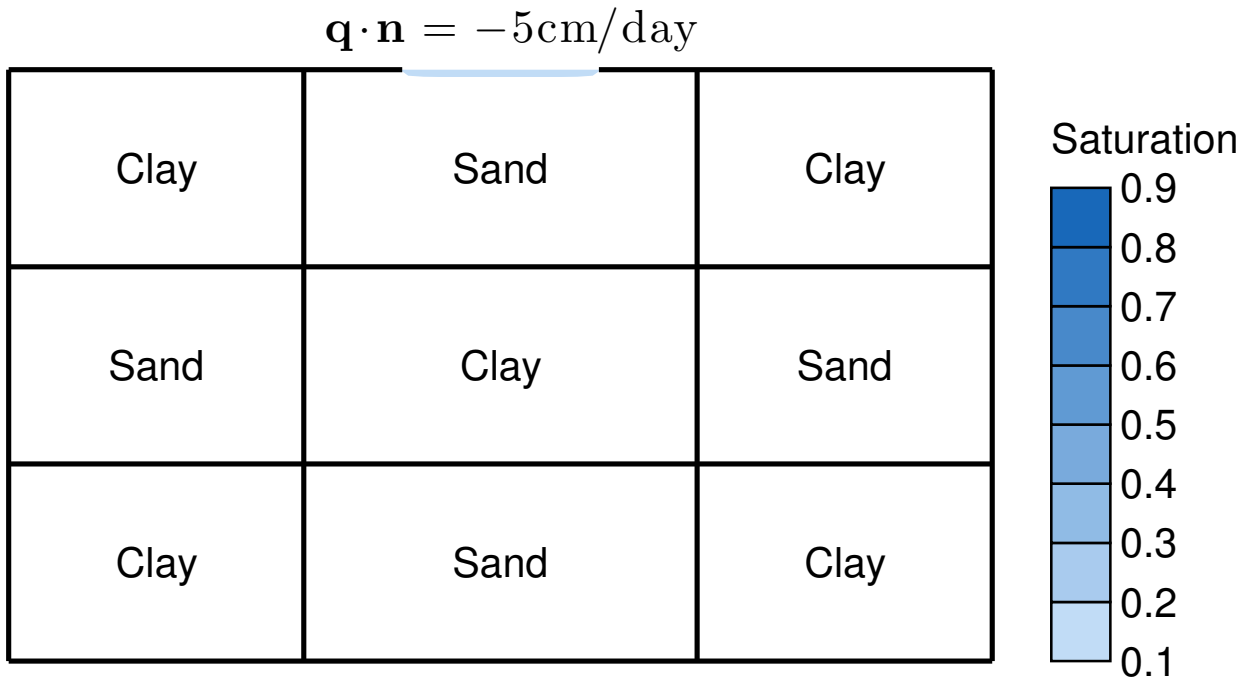

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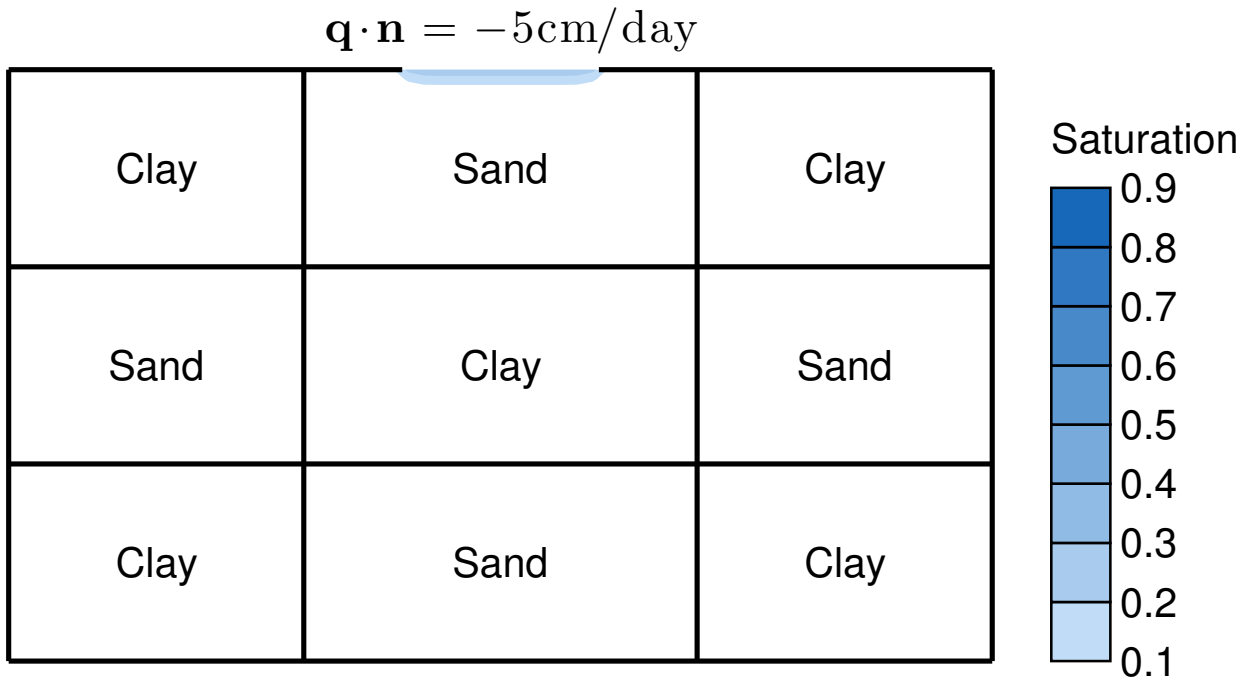

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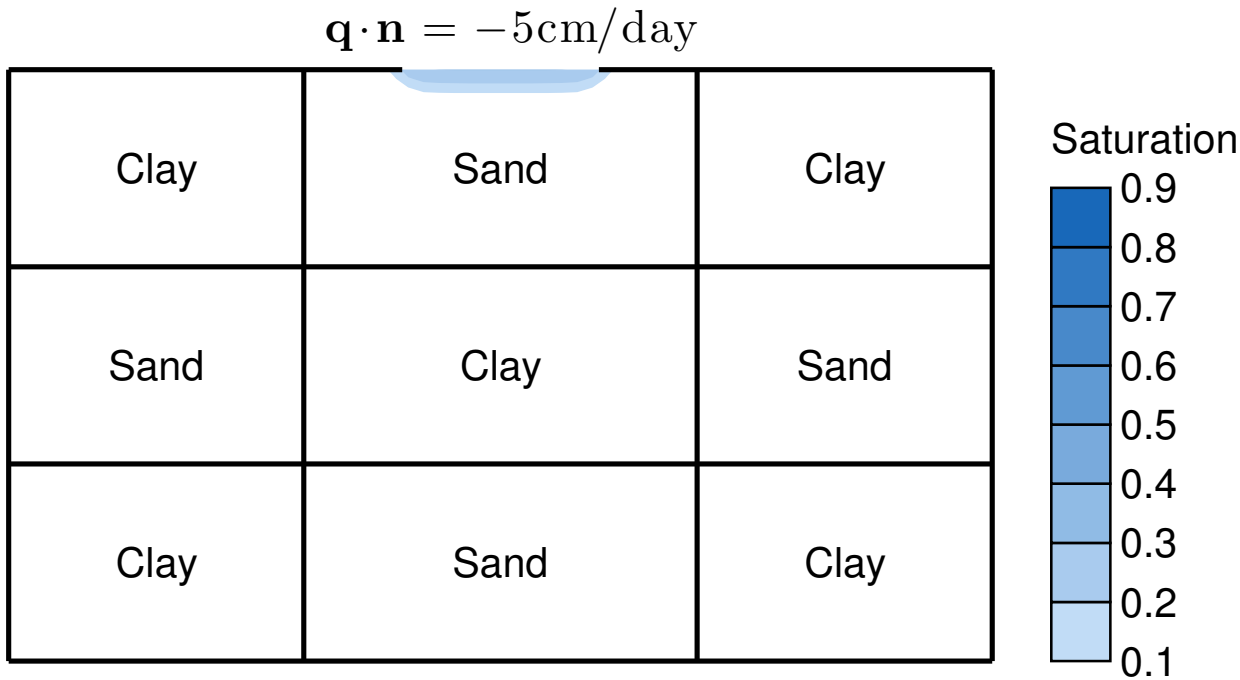

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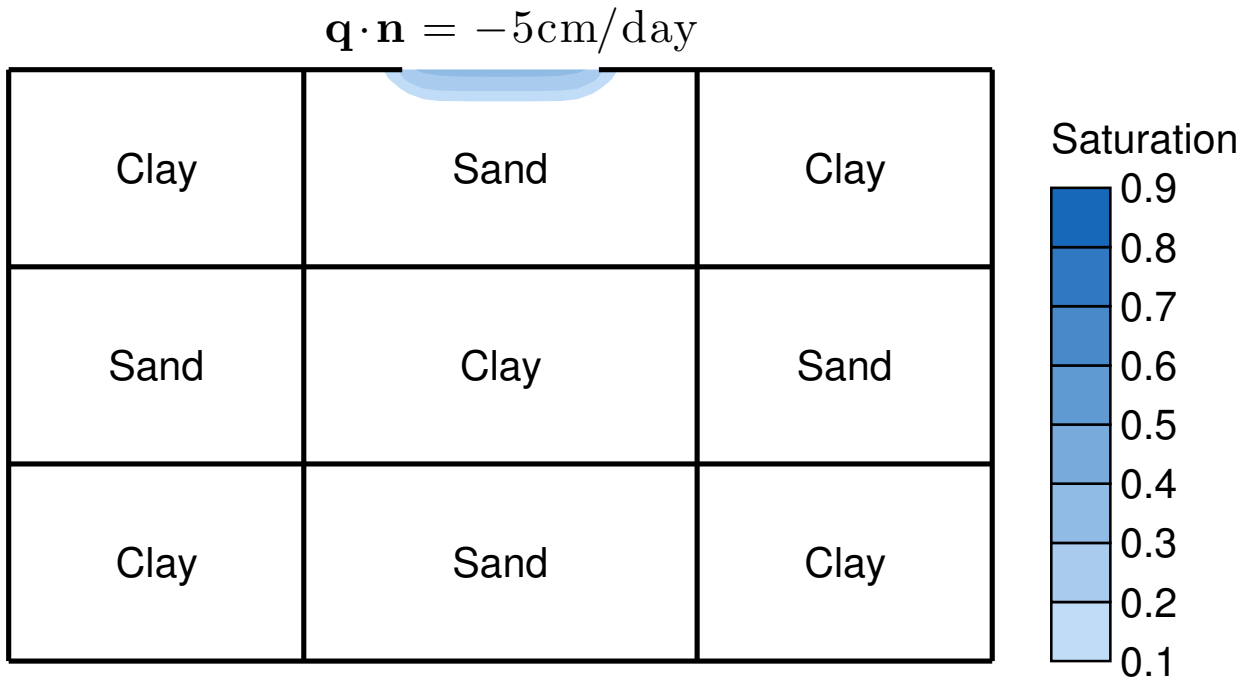

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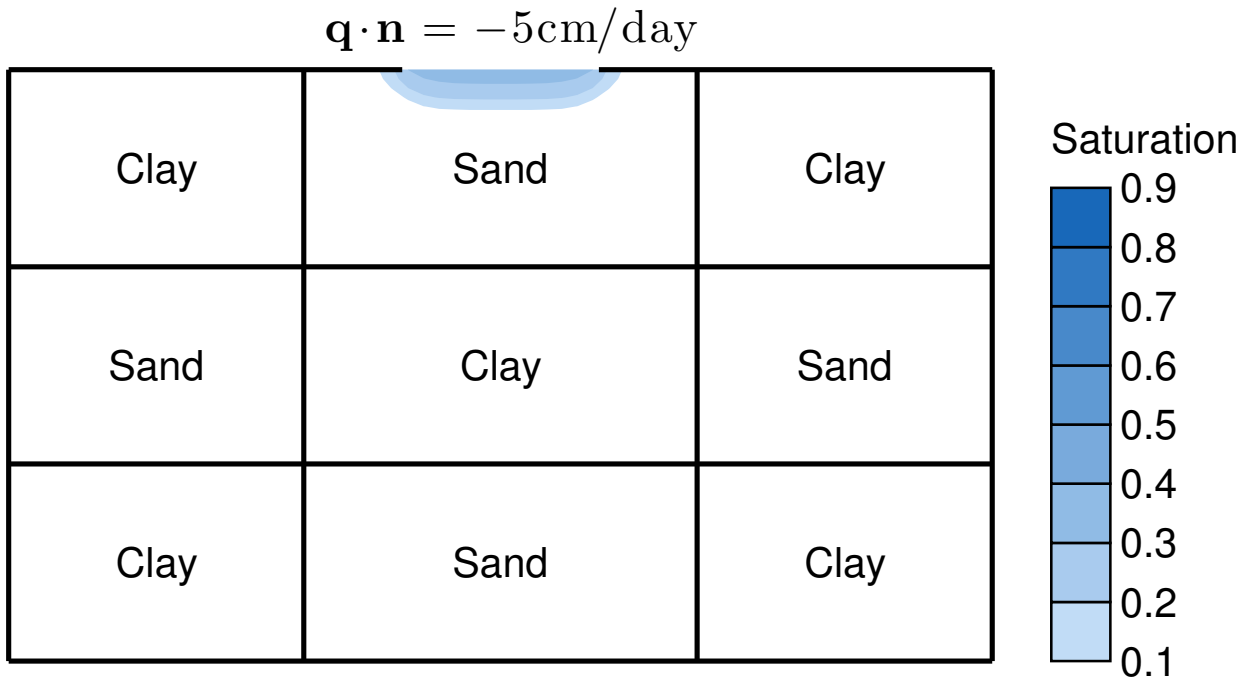

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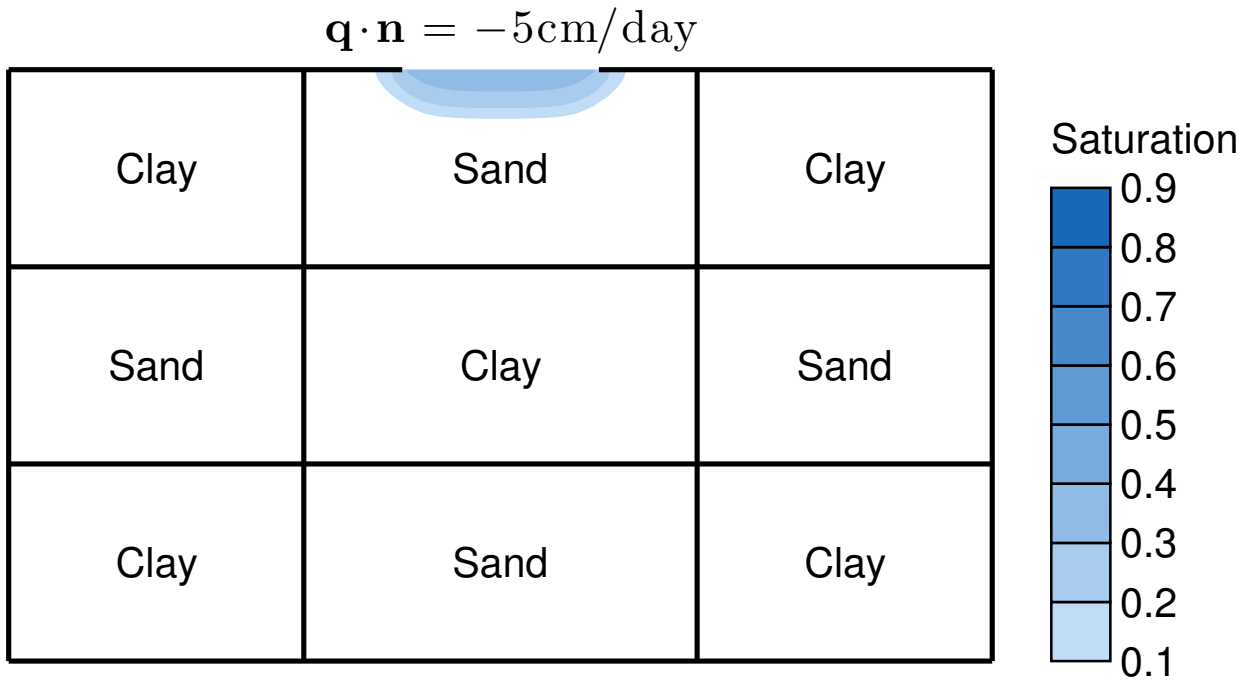

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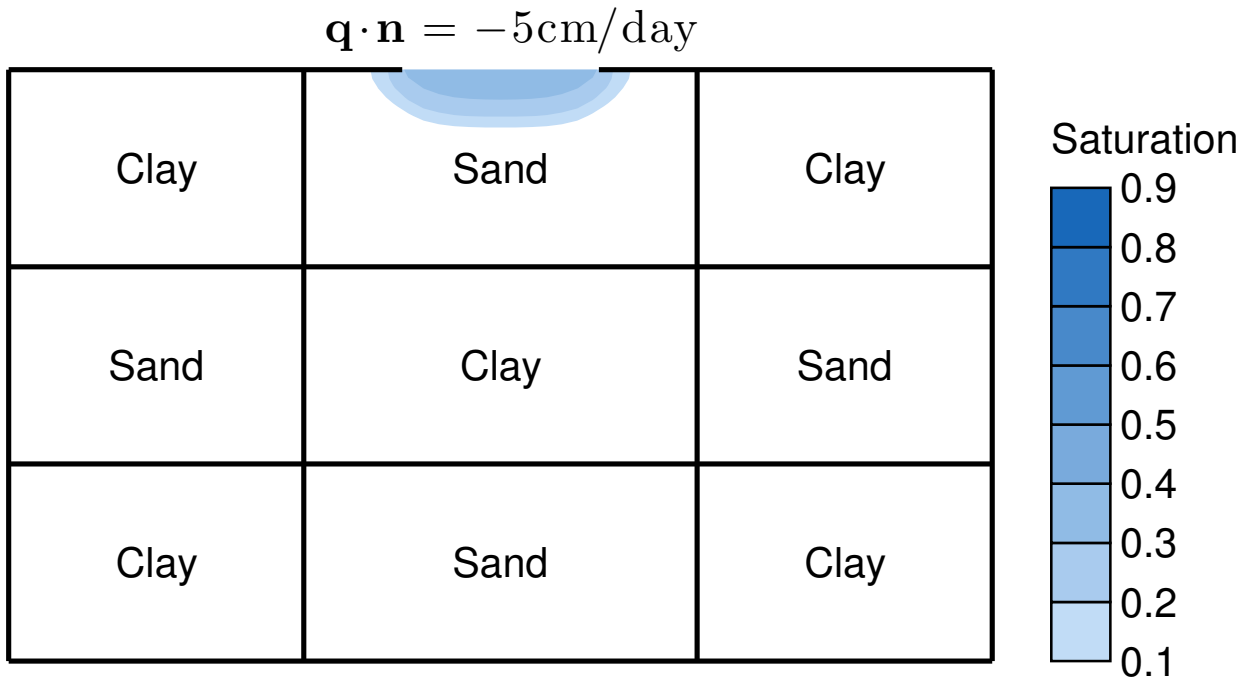

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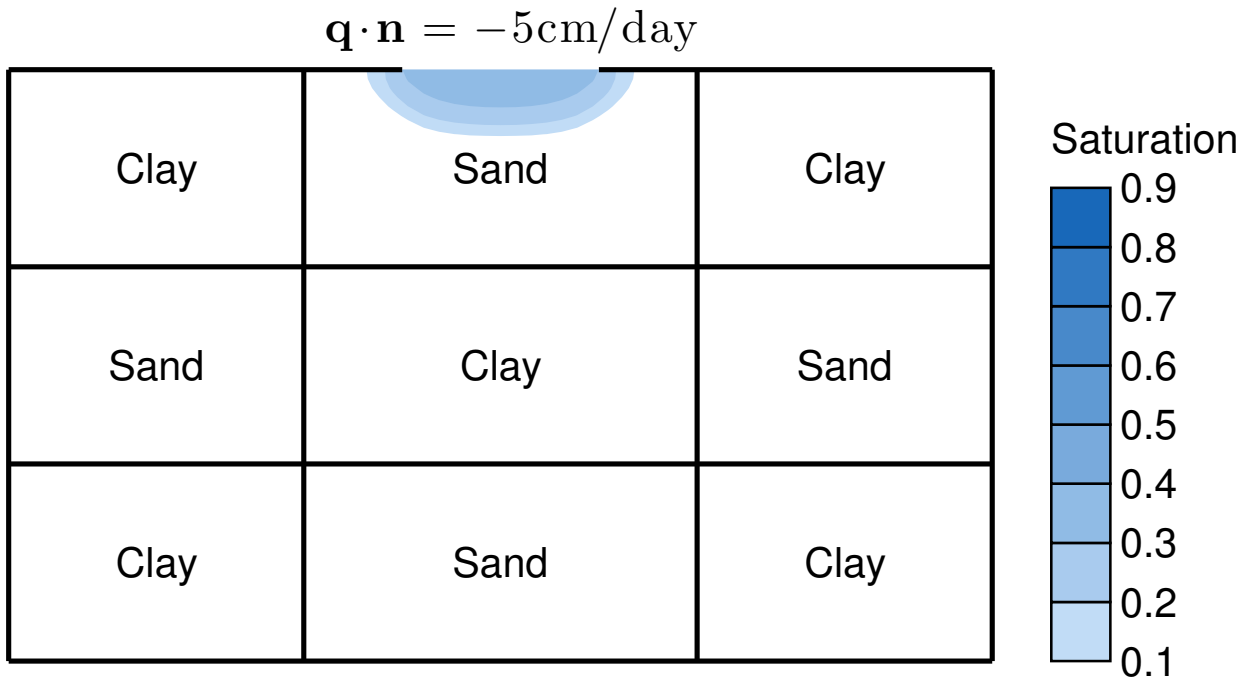

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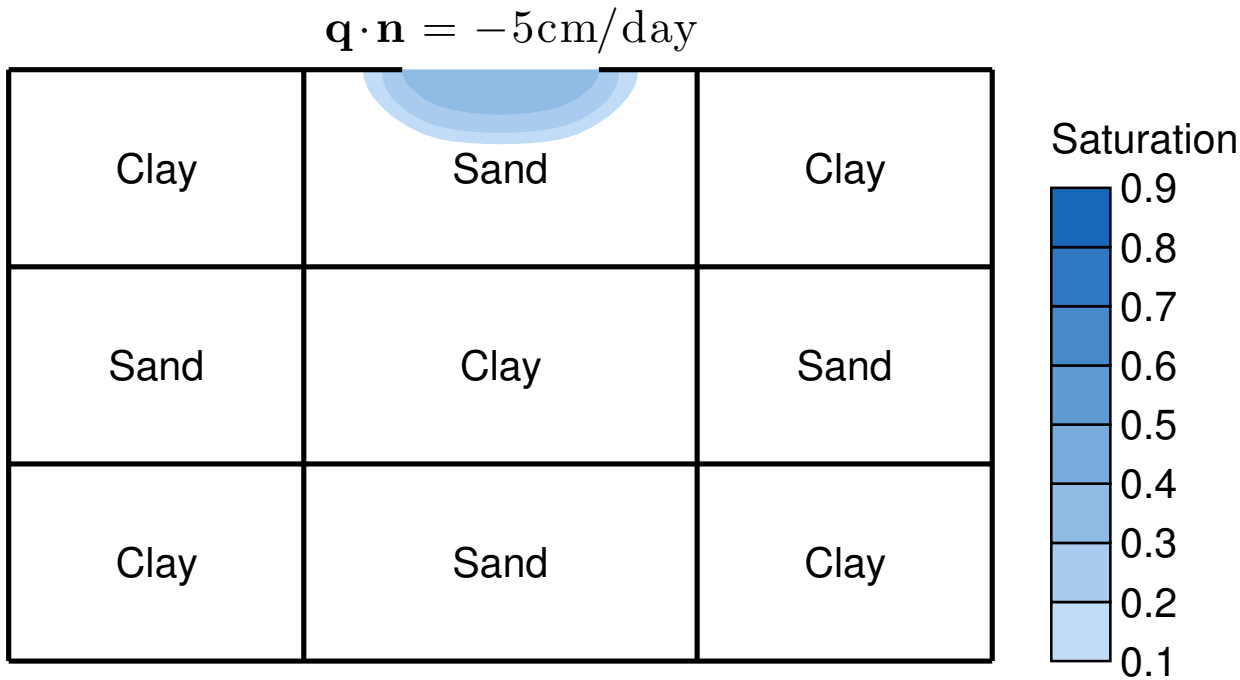

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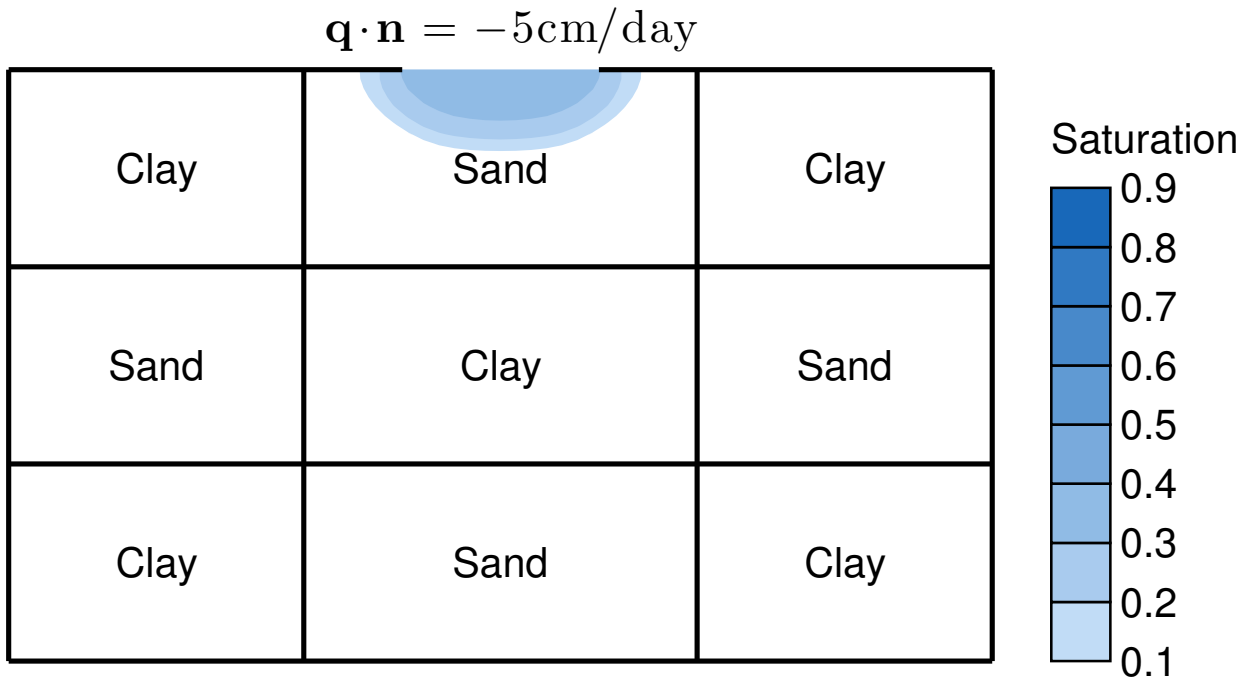

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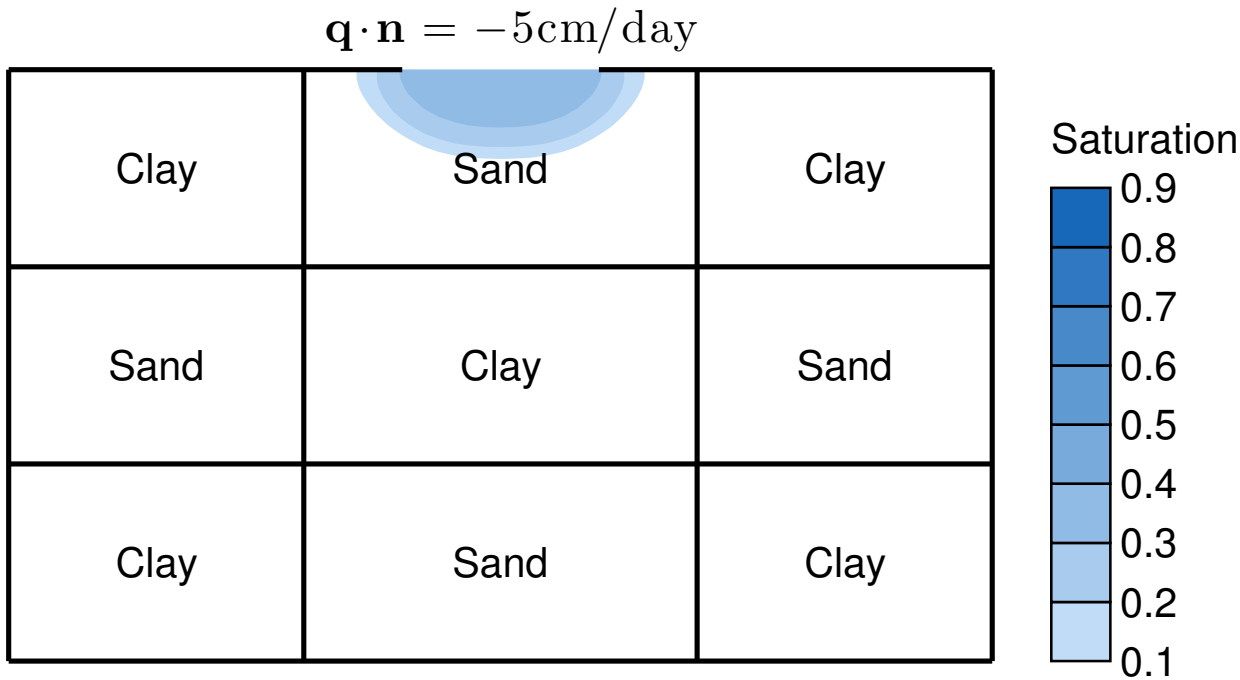

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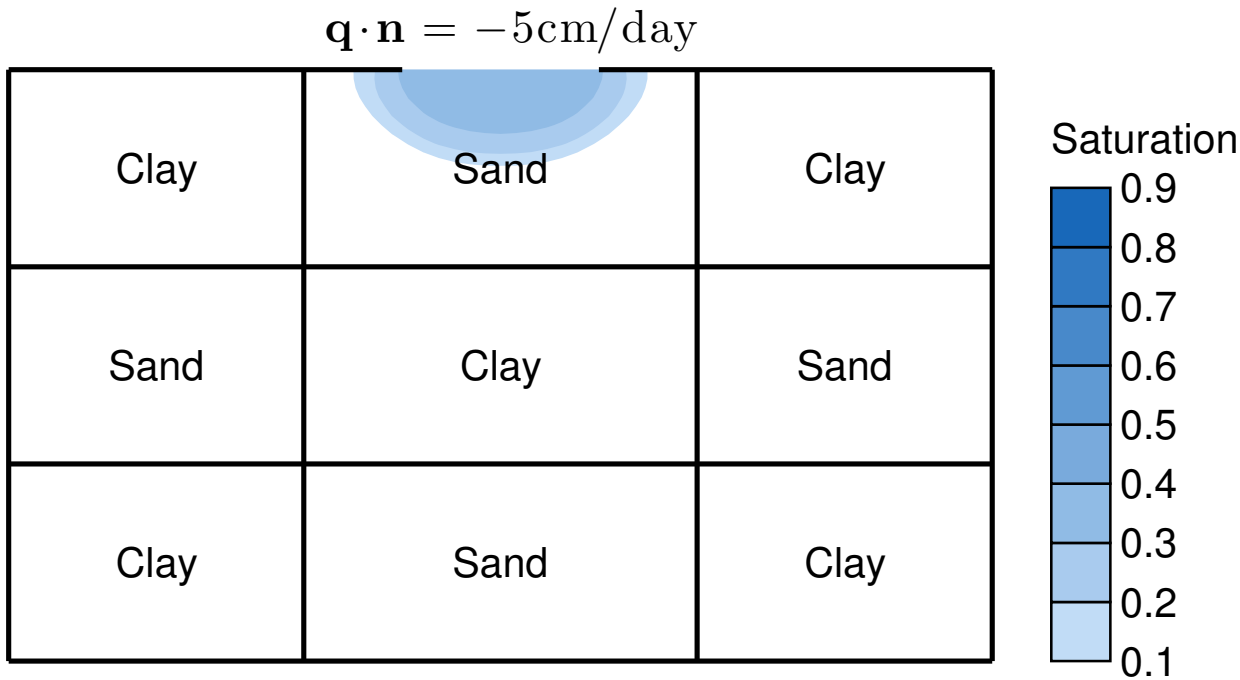

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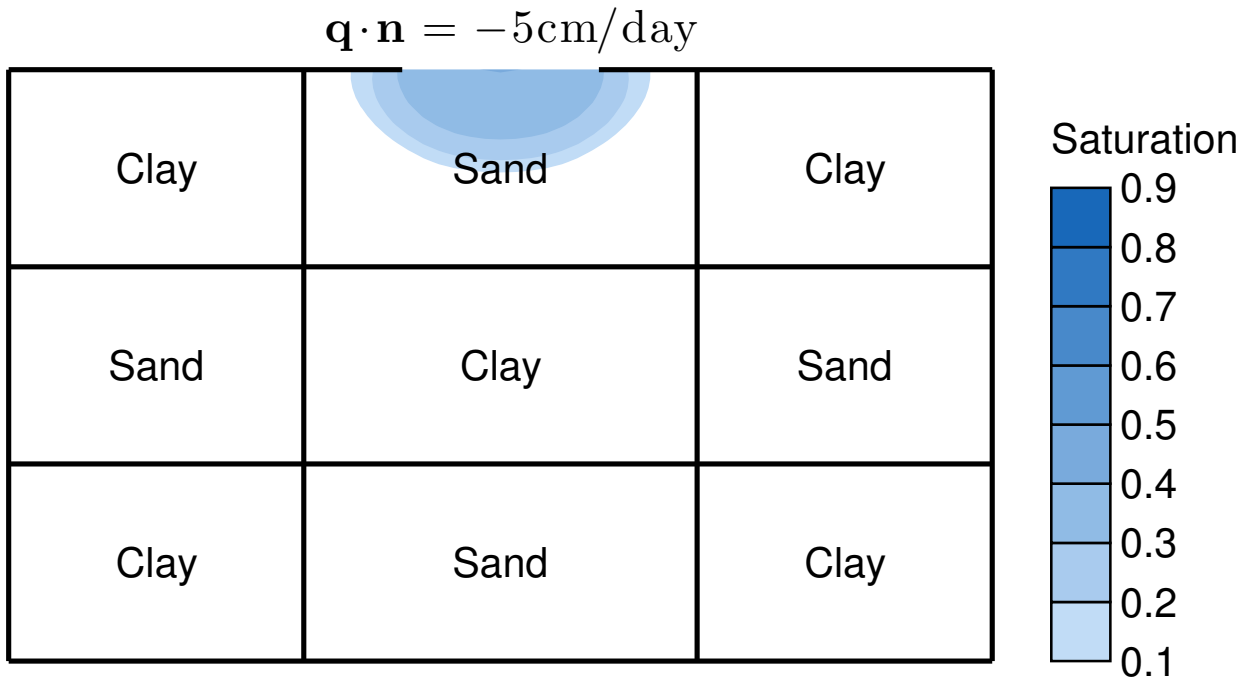

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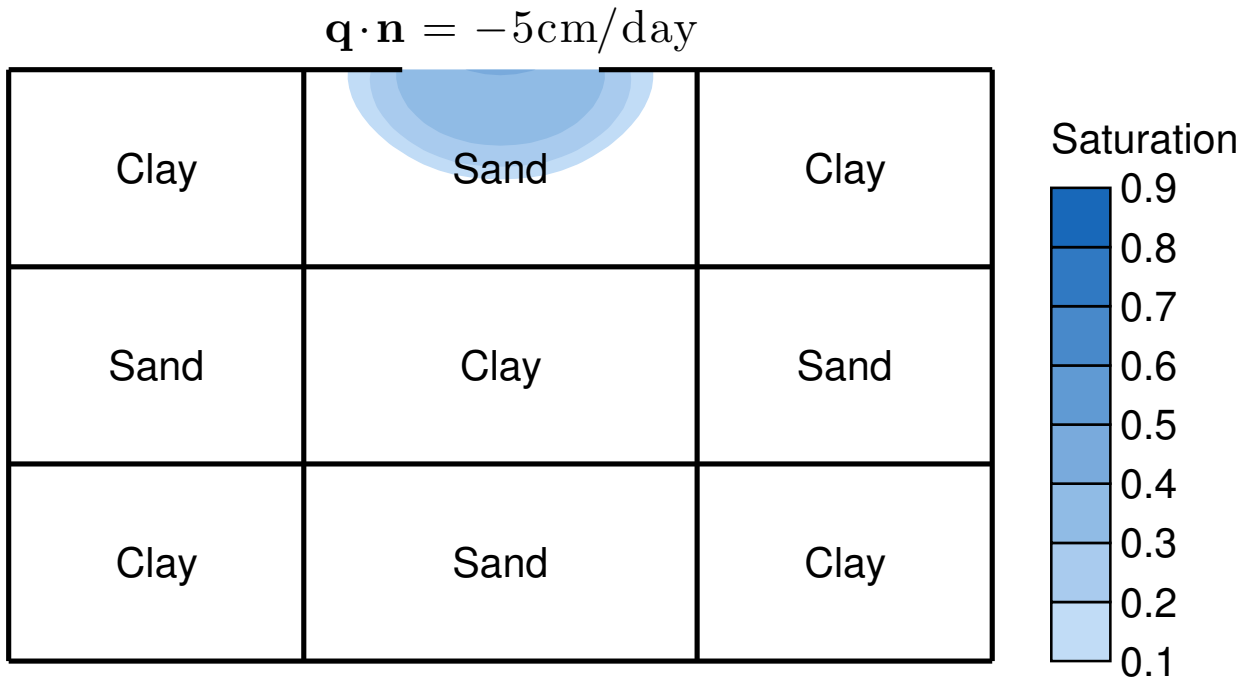

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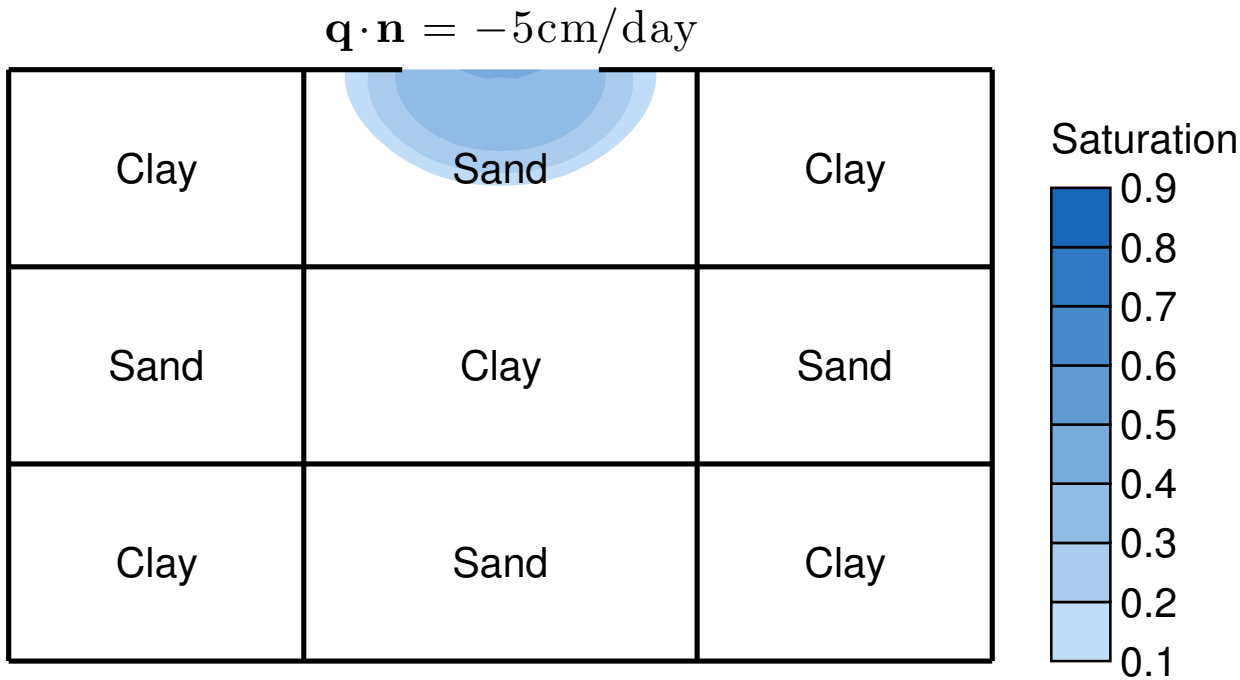

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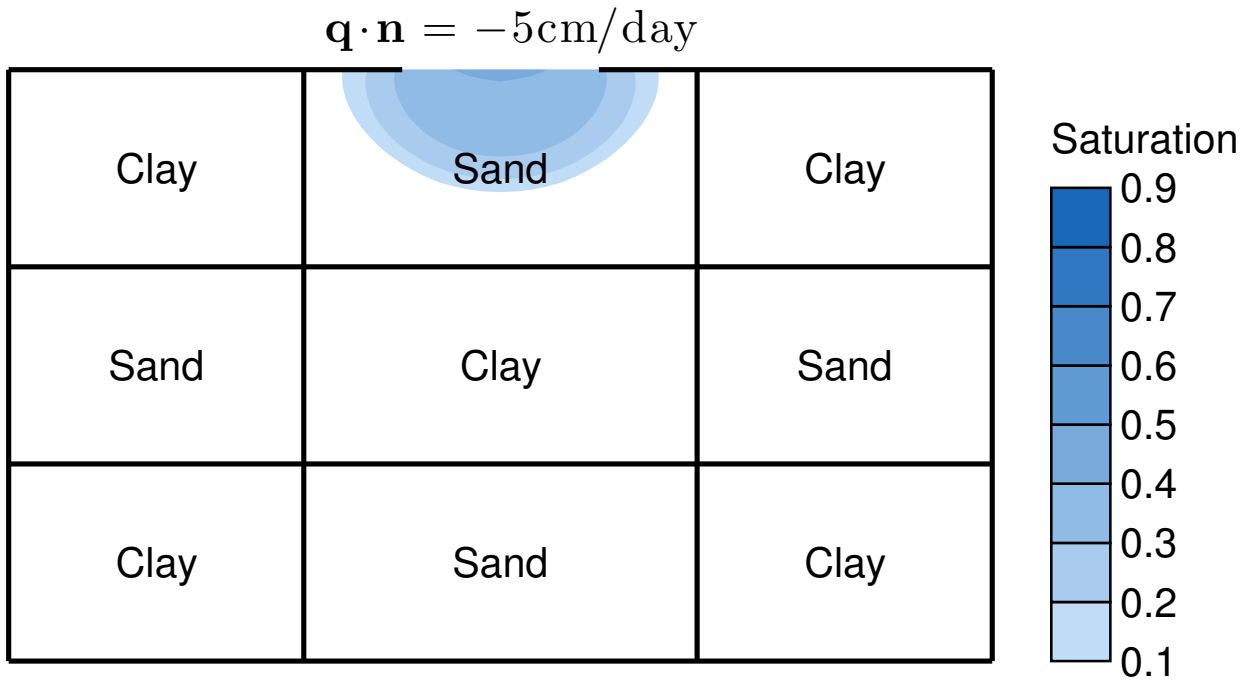

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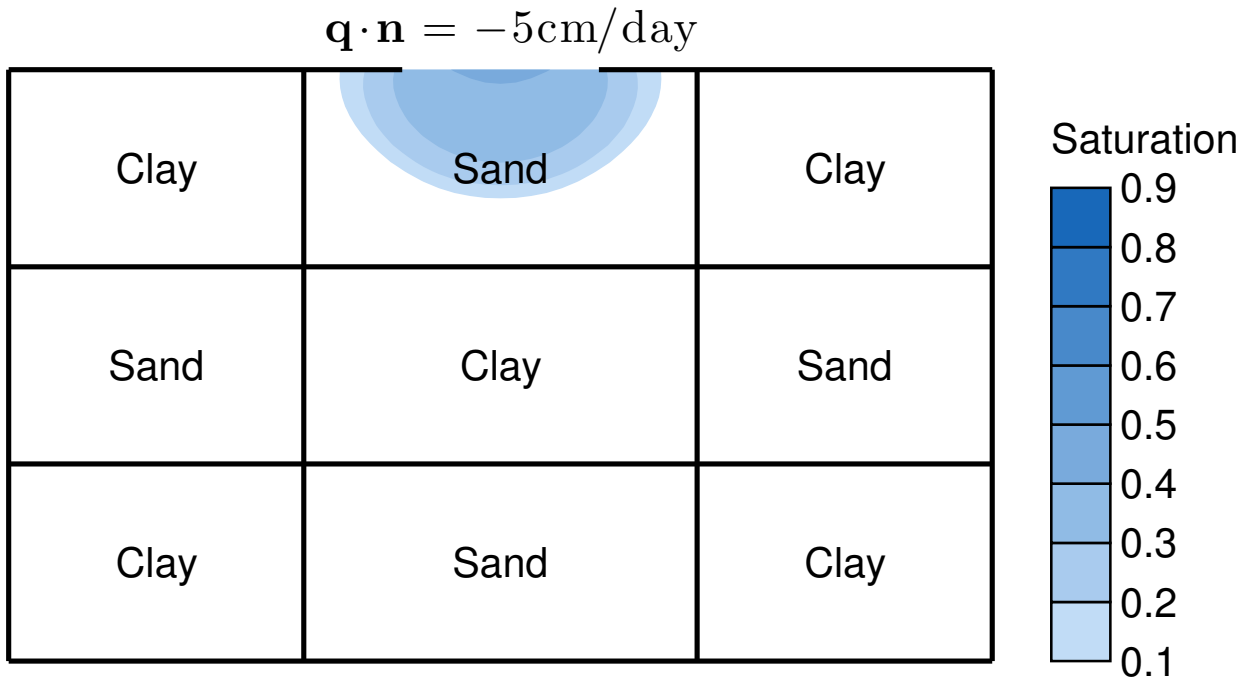

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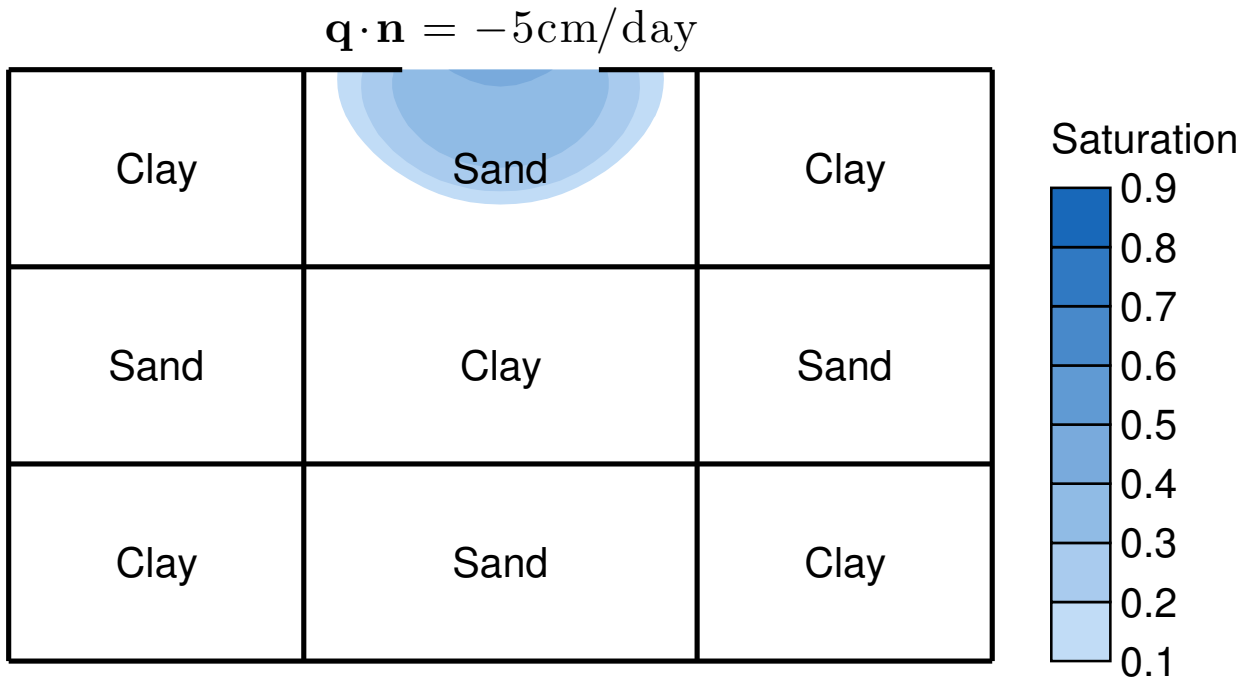

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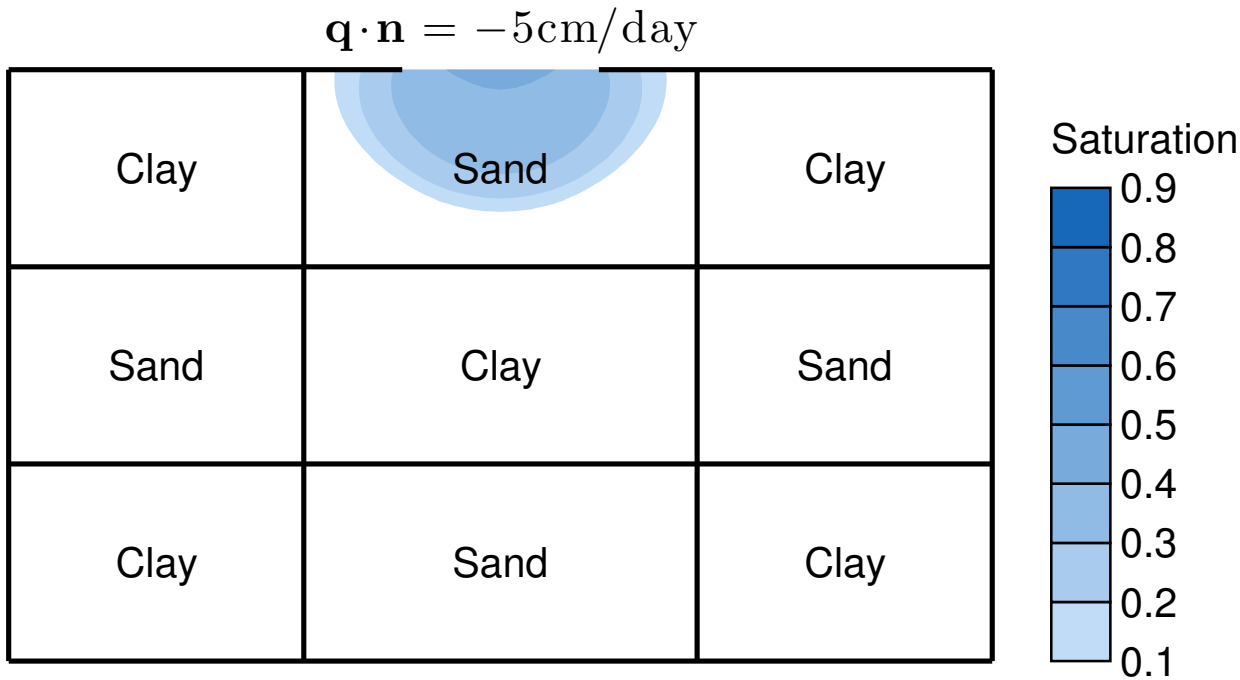

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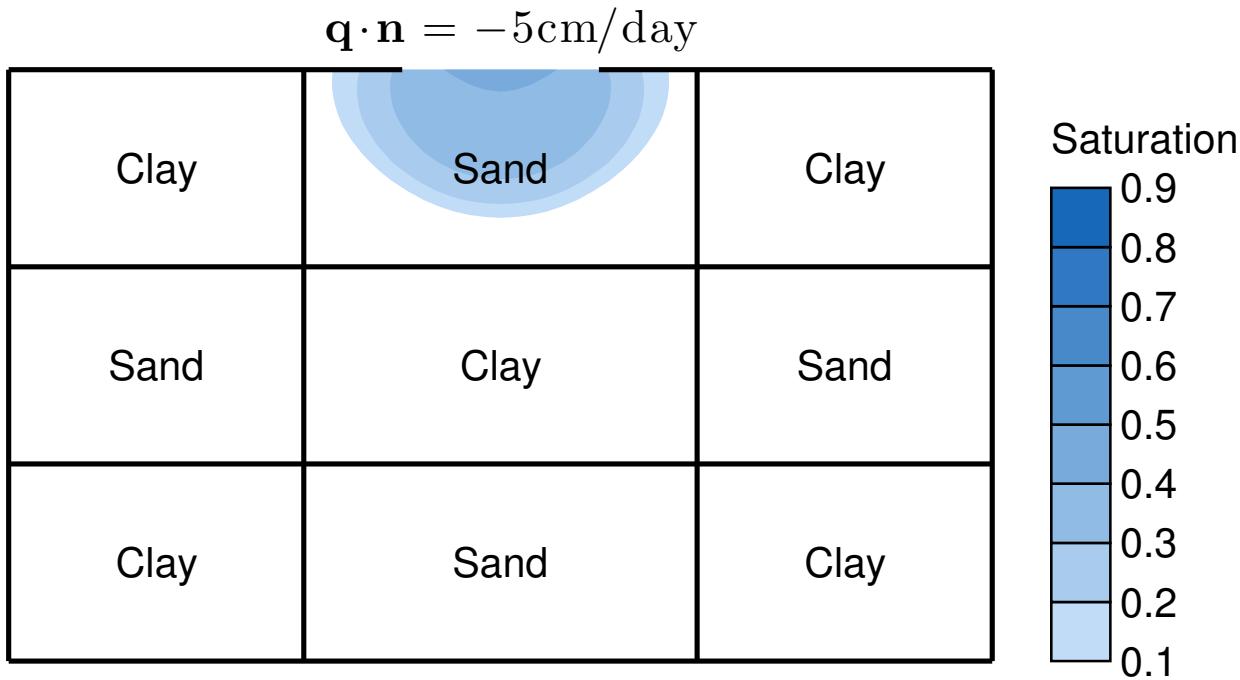

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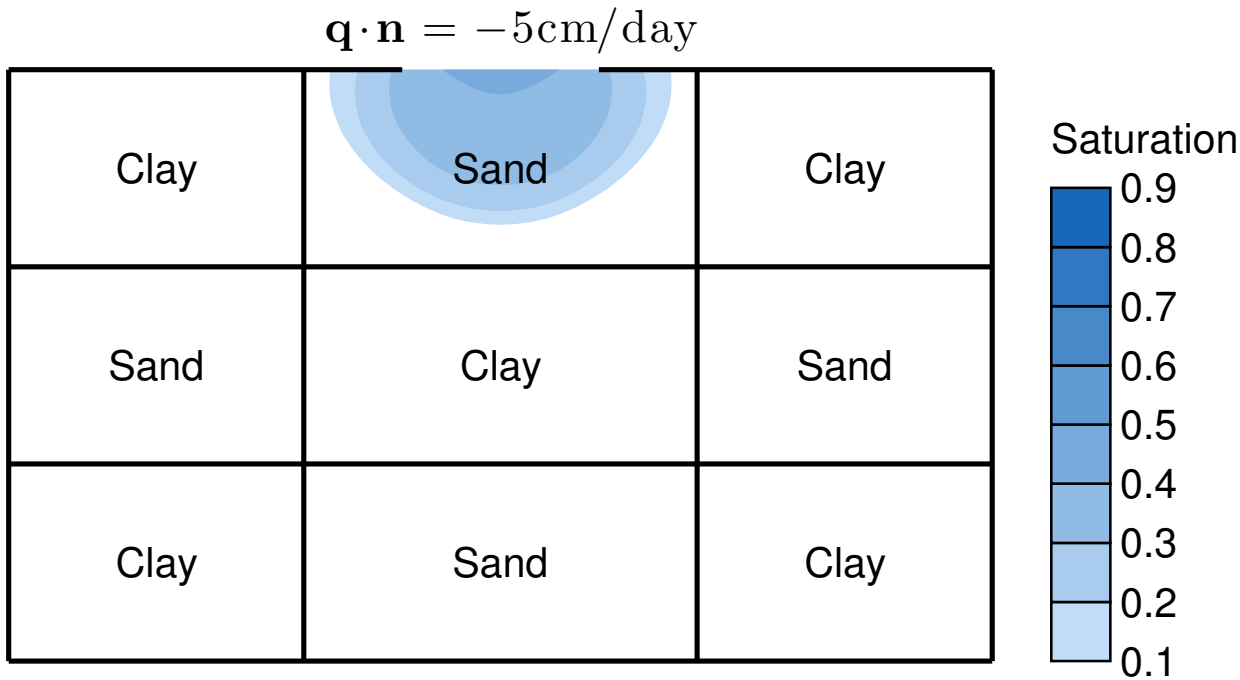

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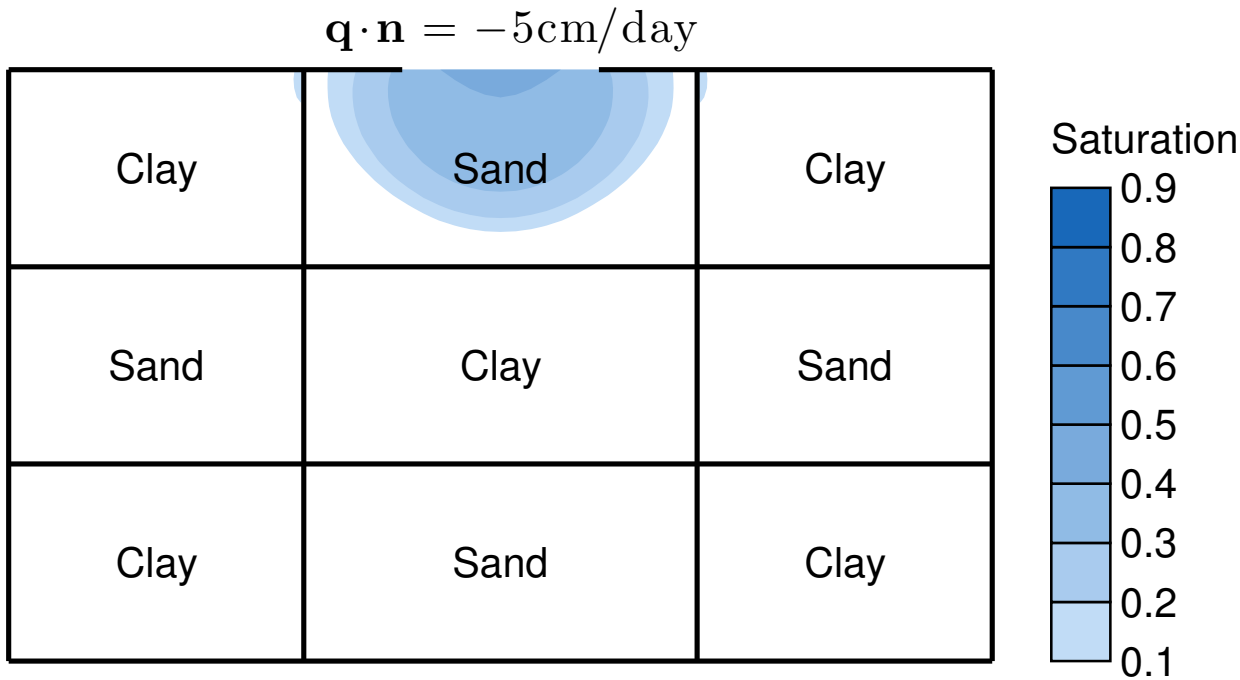

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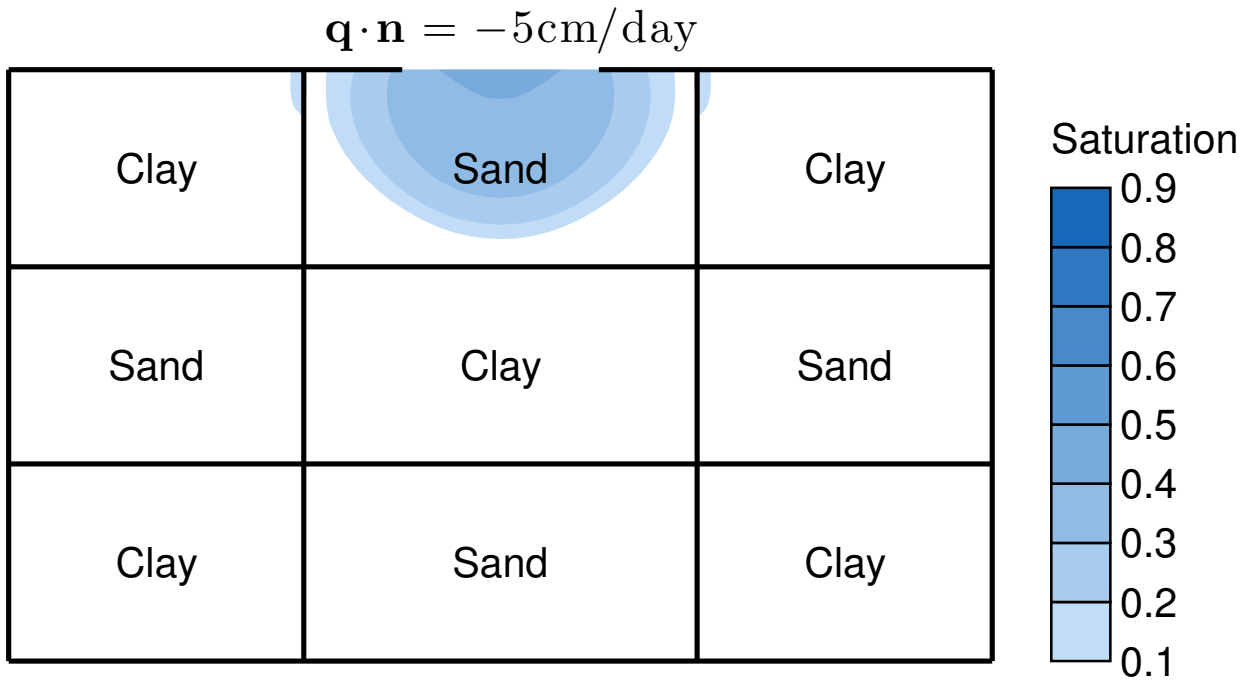

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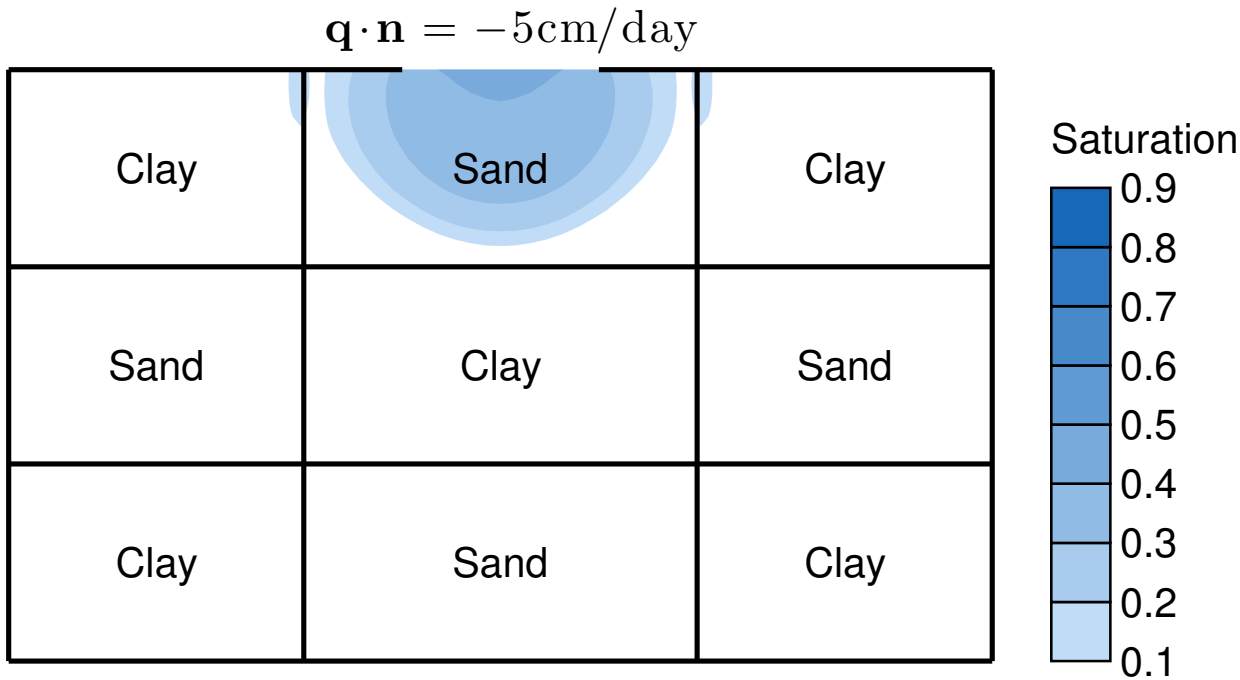

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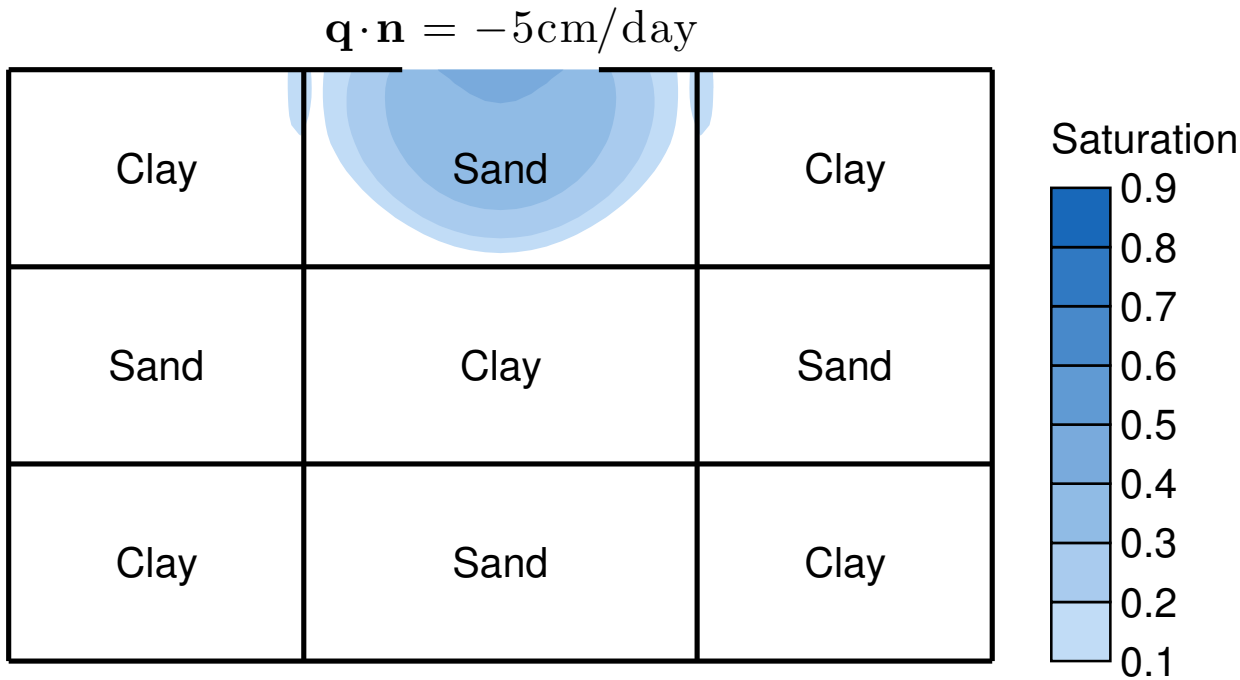

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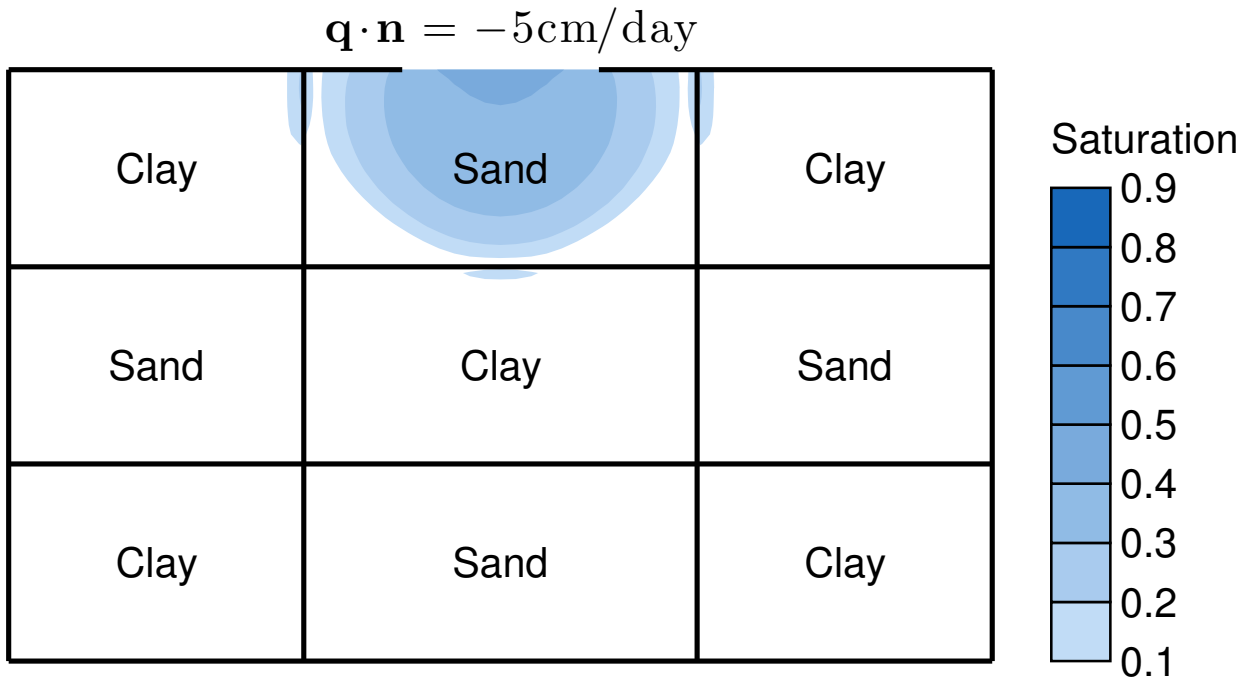

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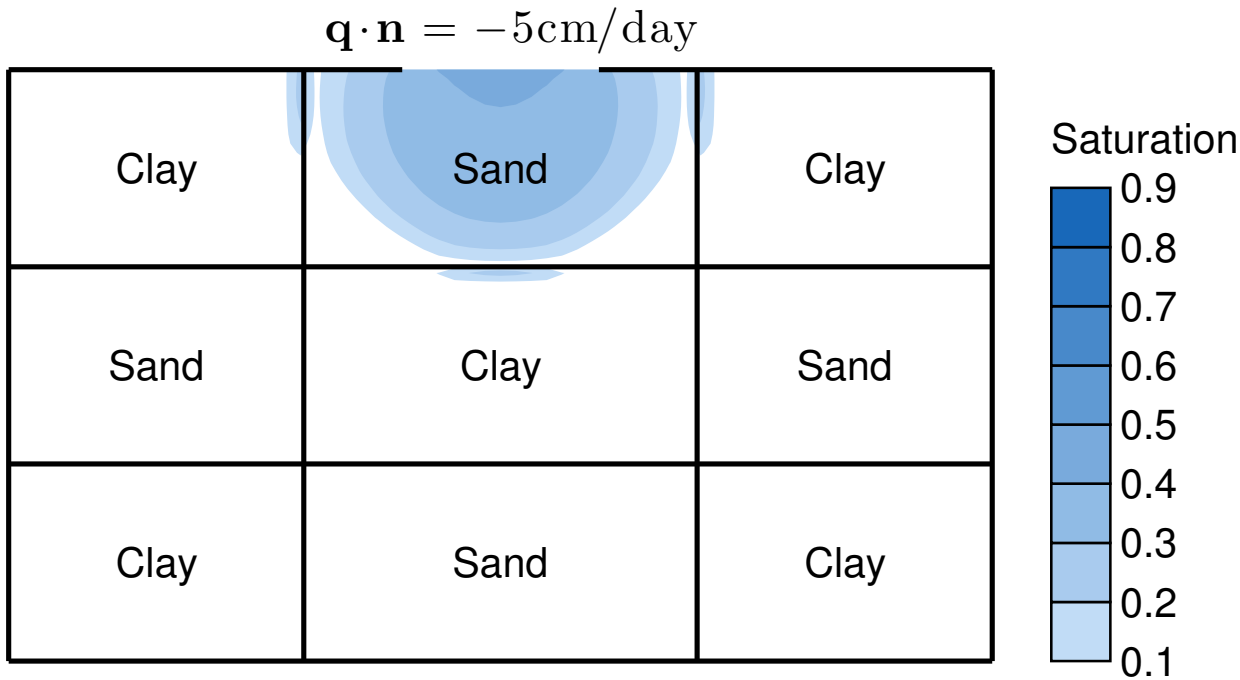

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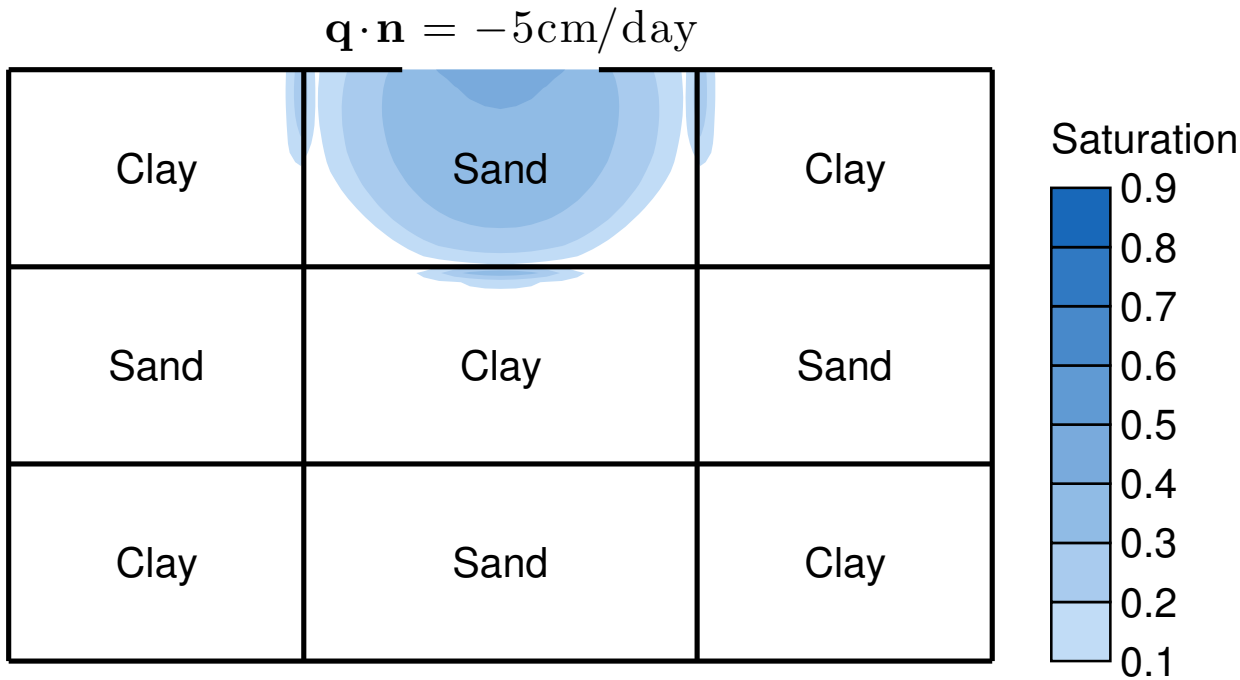
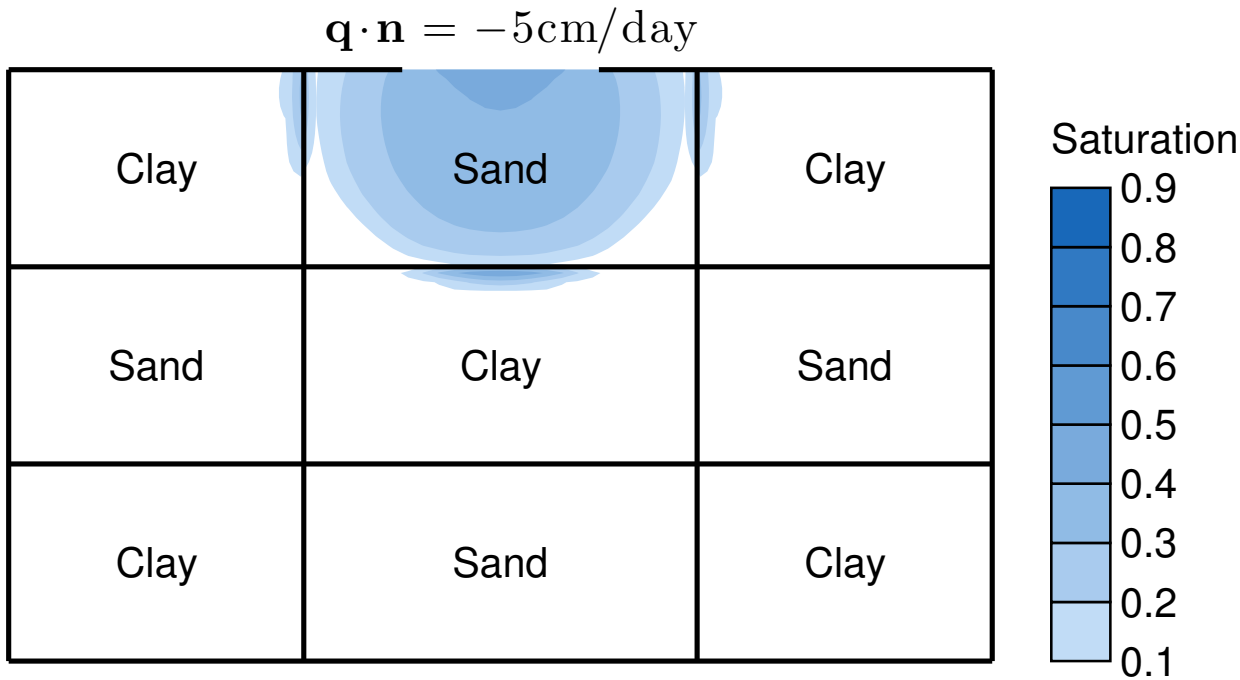
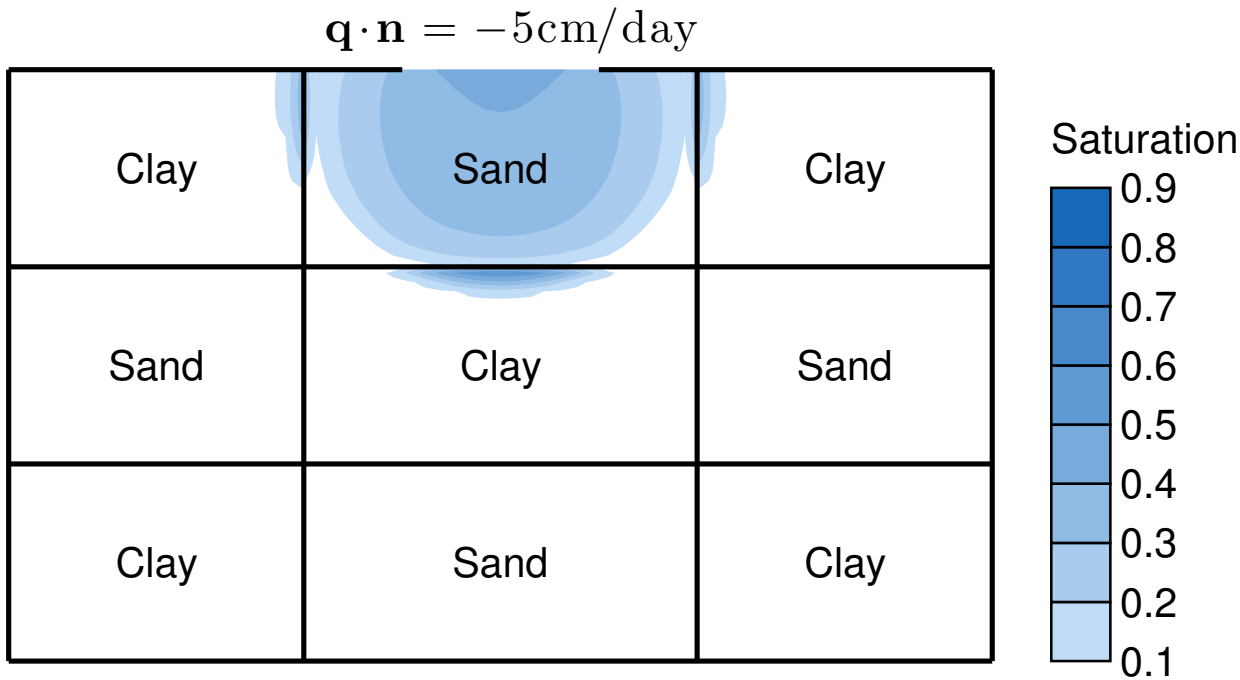


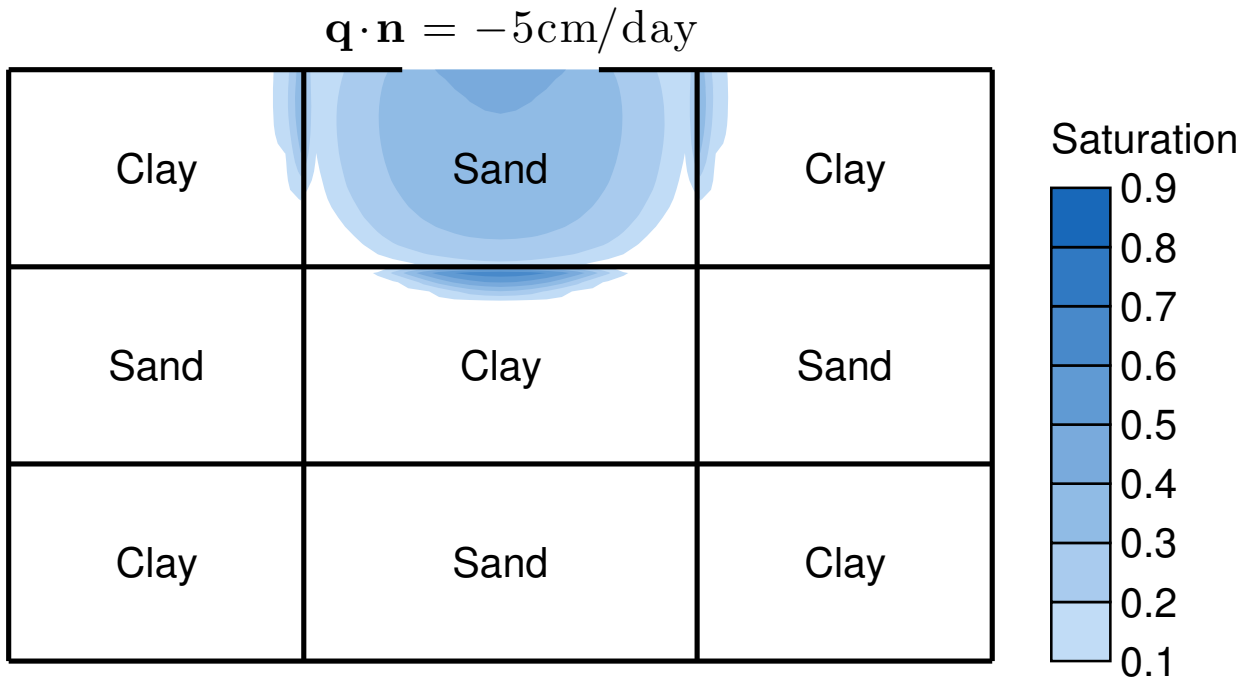
Test Problem: Water infiltration



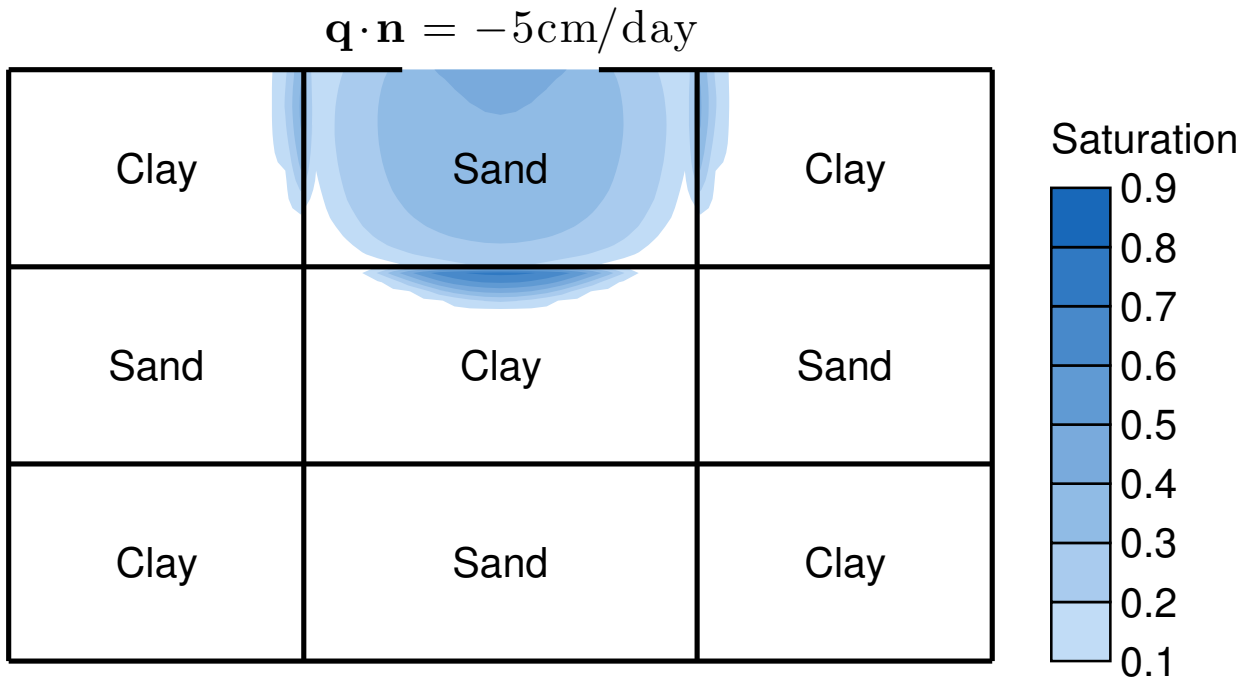
Test Problem: Water infiltration



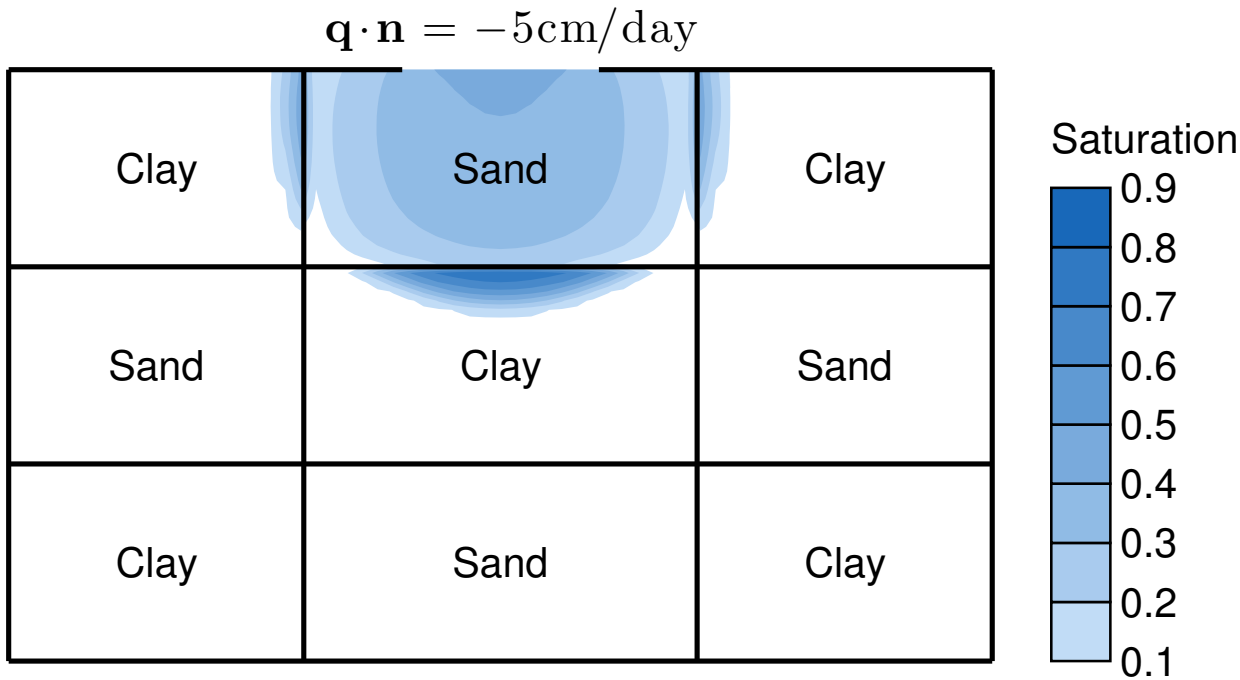
Test Problem: Water infiltration



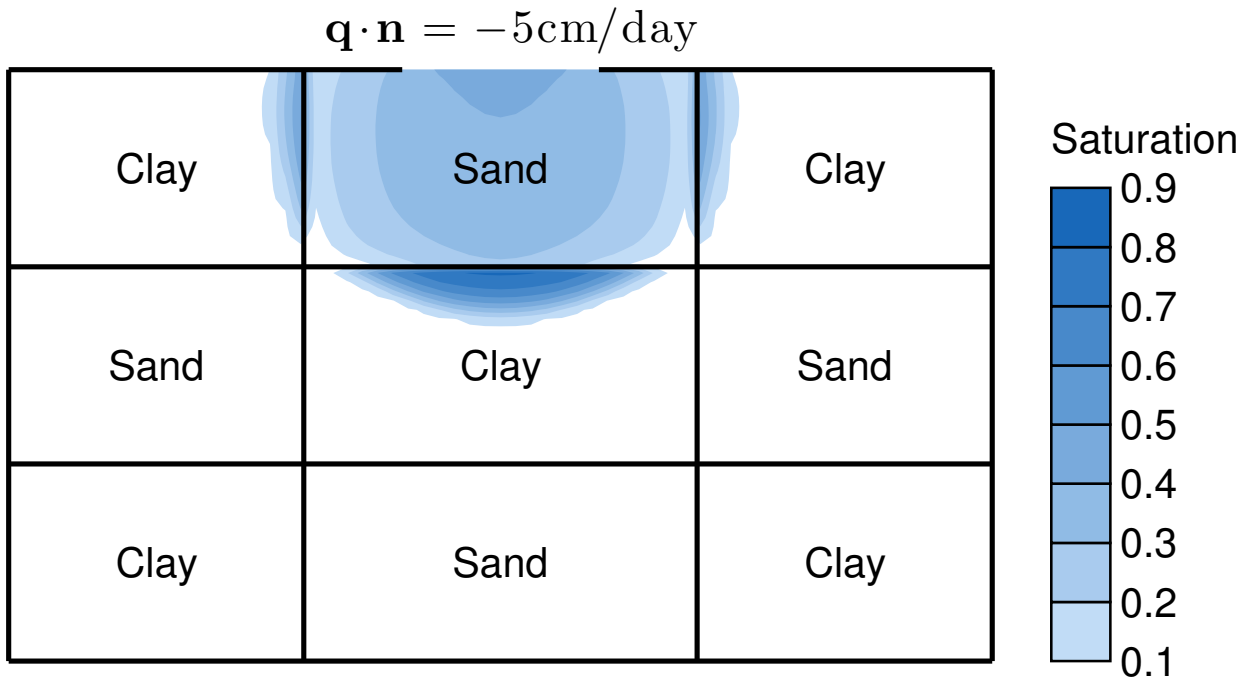
Test Problem: Water infiltration



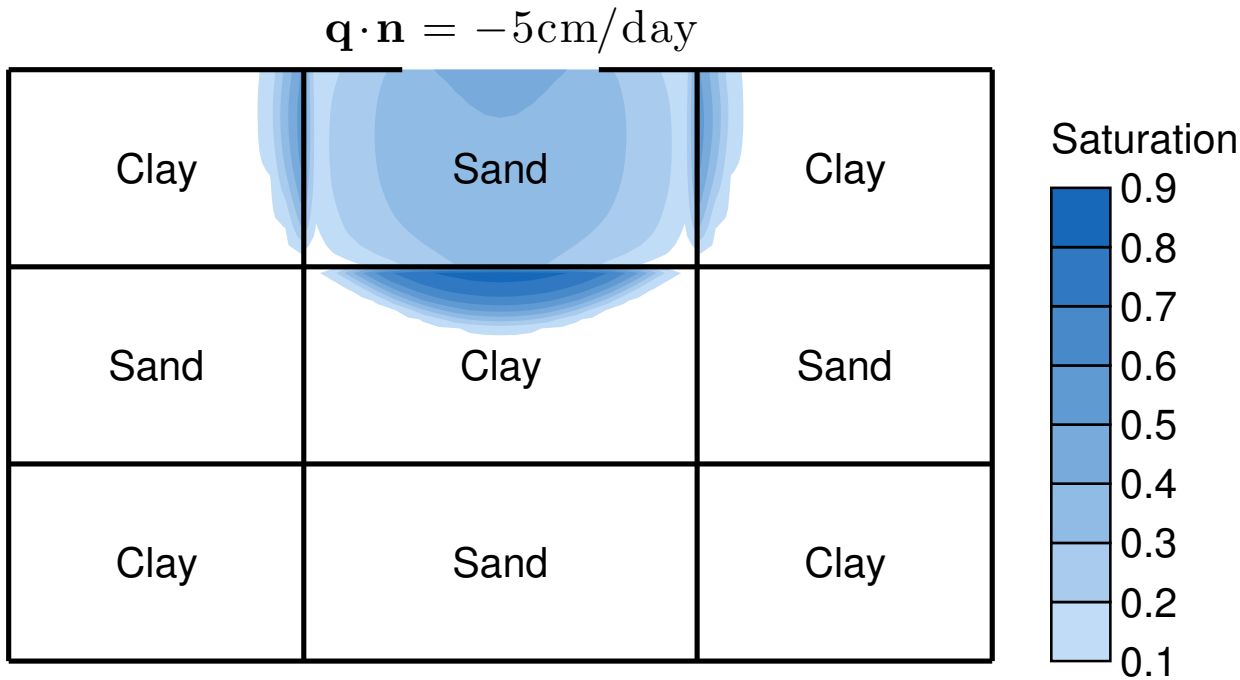
Test Problem: Water infiltration



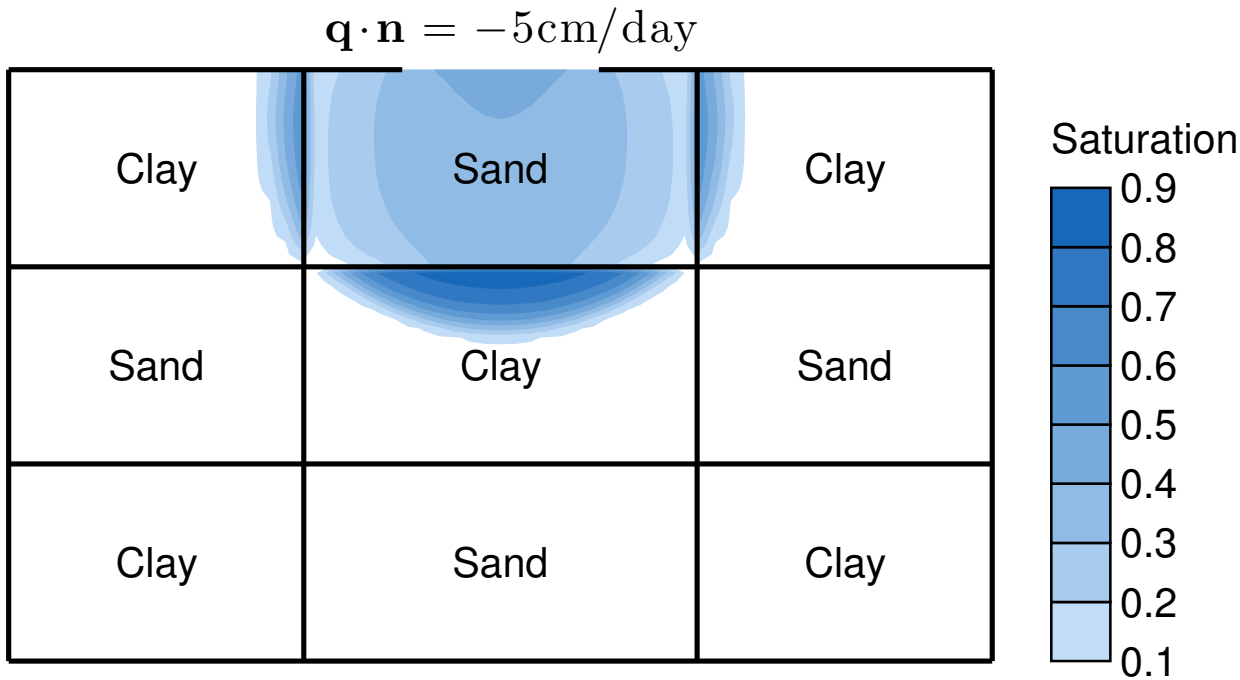
Test Problem: Water infiltration



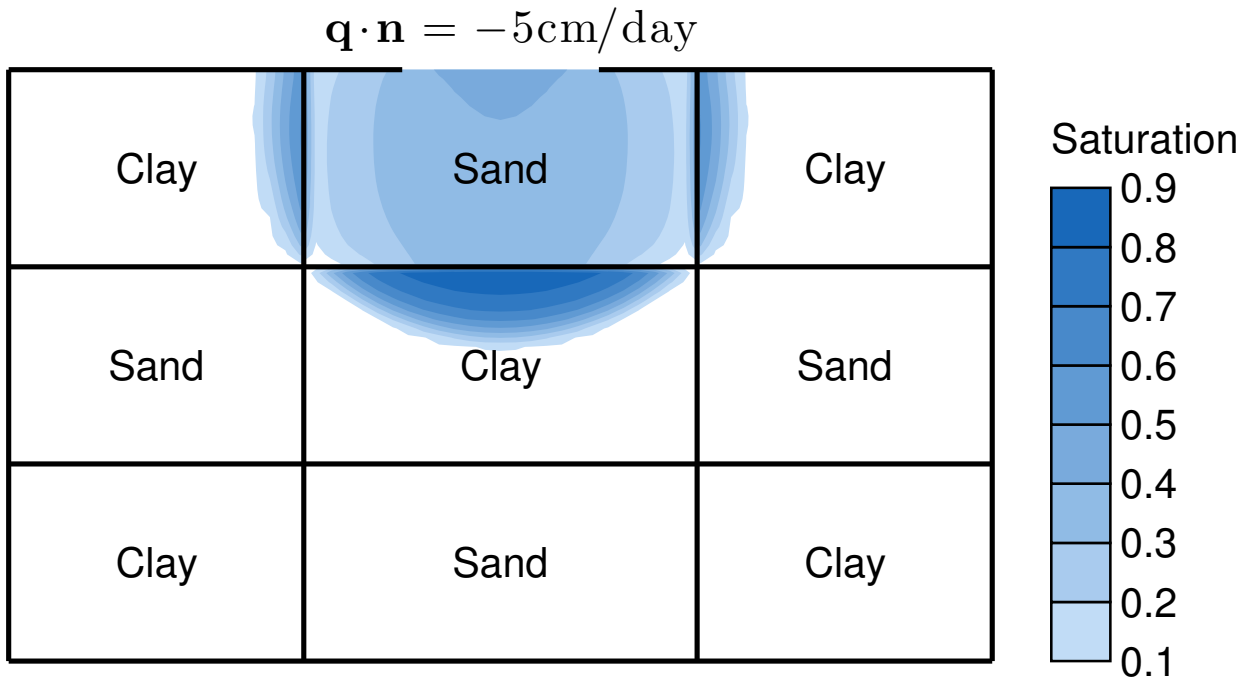
Test Problem: Water infiltration



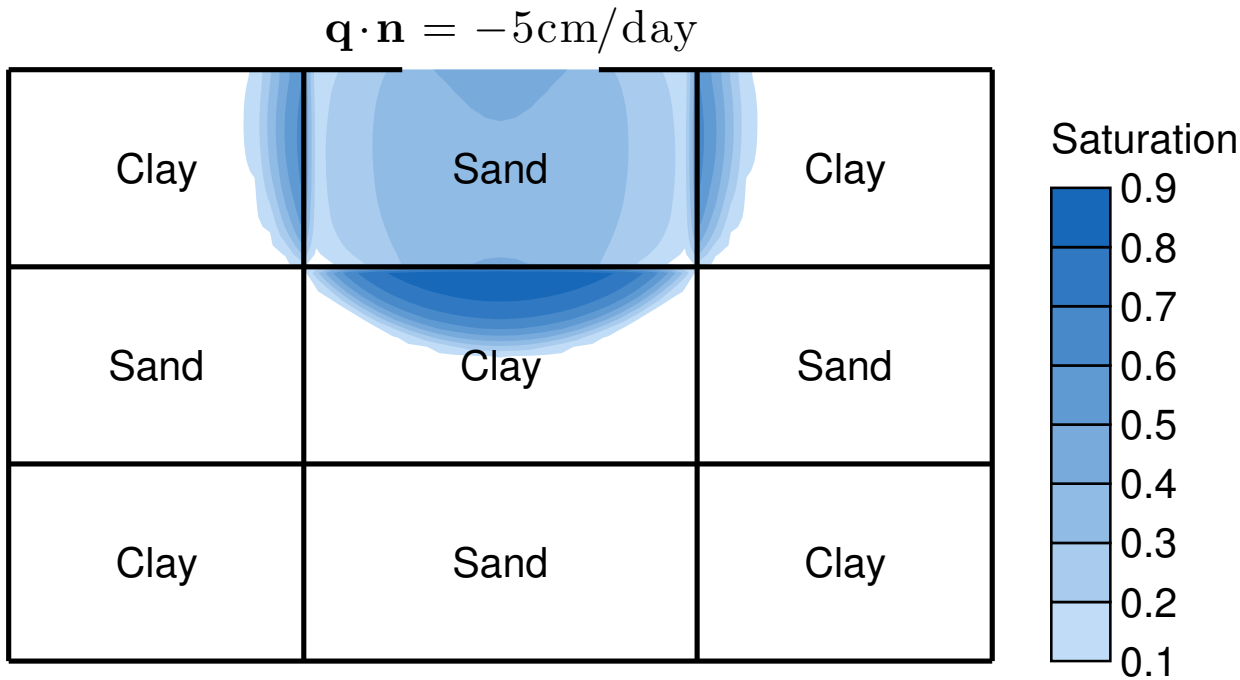
Test Problem: Water infiltration



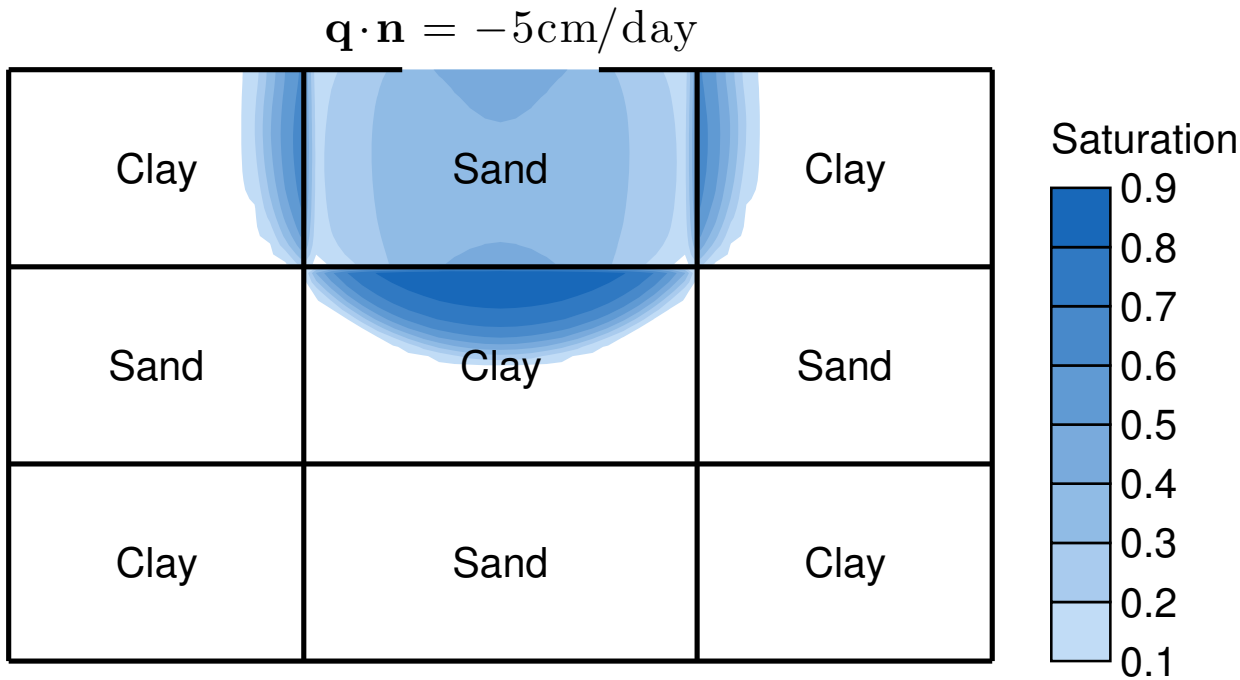
Test Problem: Water infiltration



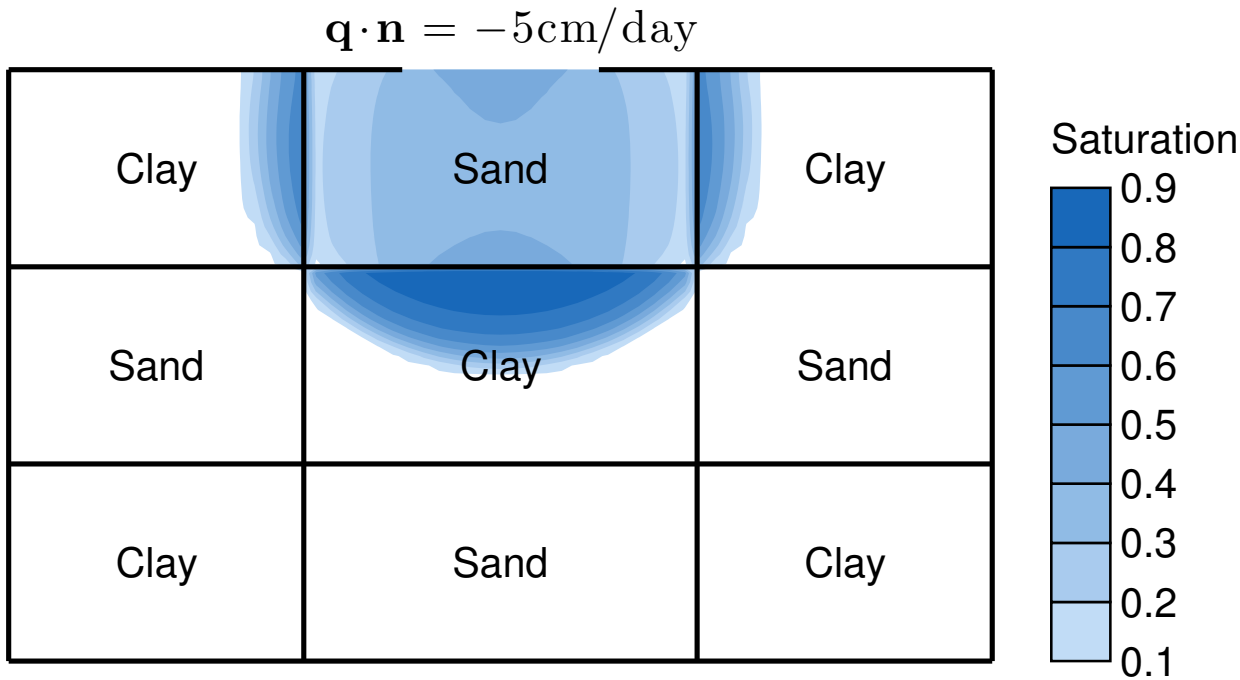
Test Problem: Water infiltration



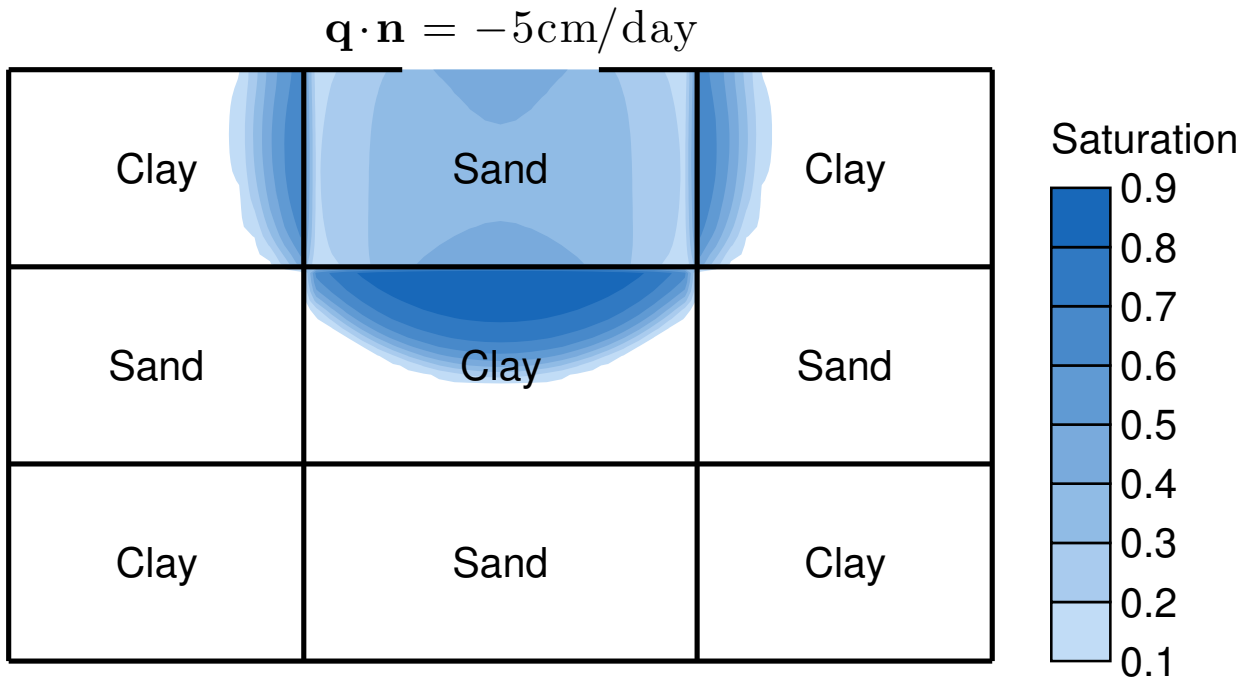
Test Problem: Water infiltration



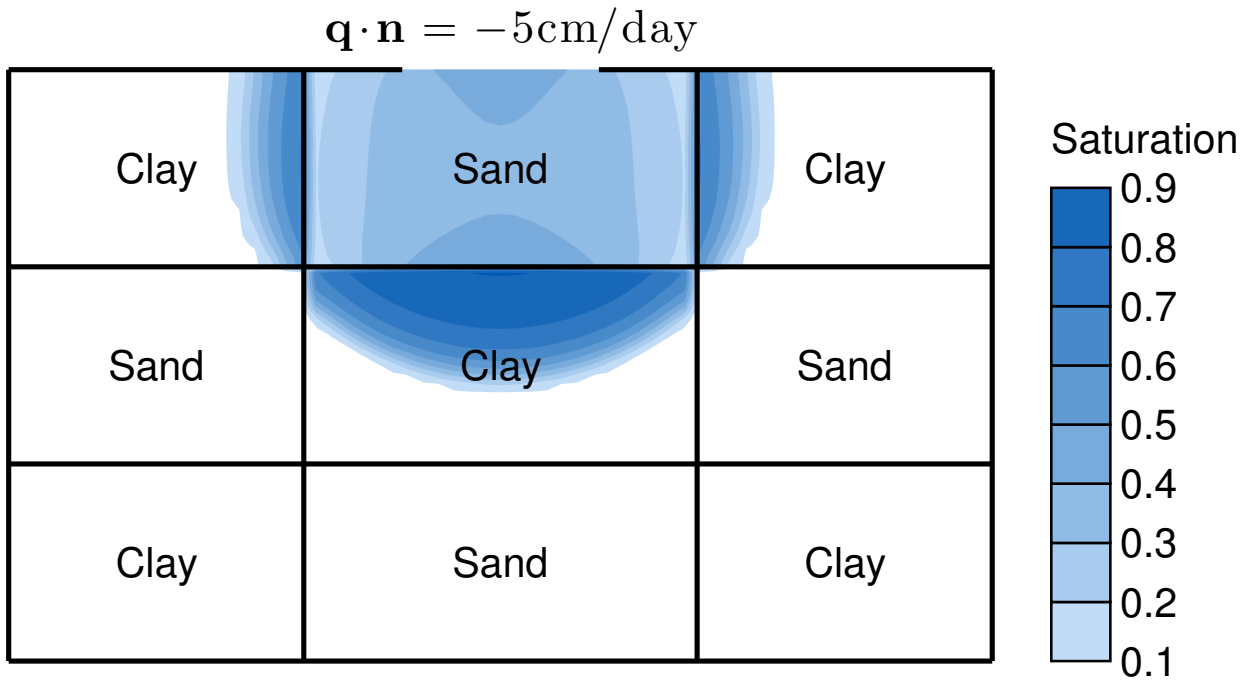
Test Problem: Water infiltration



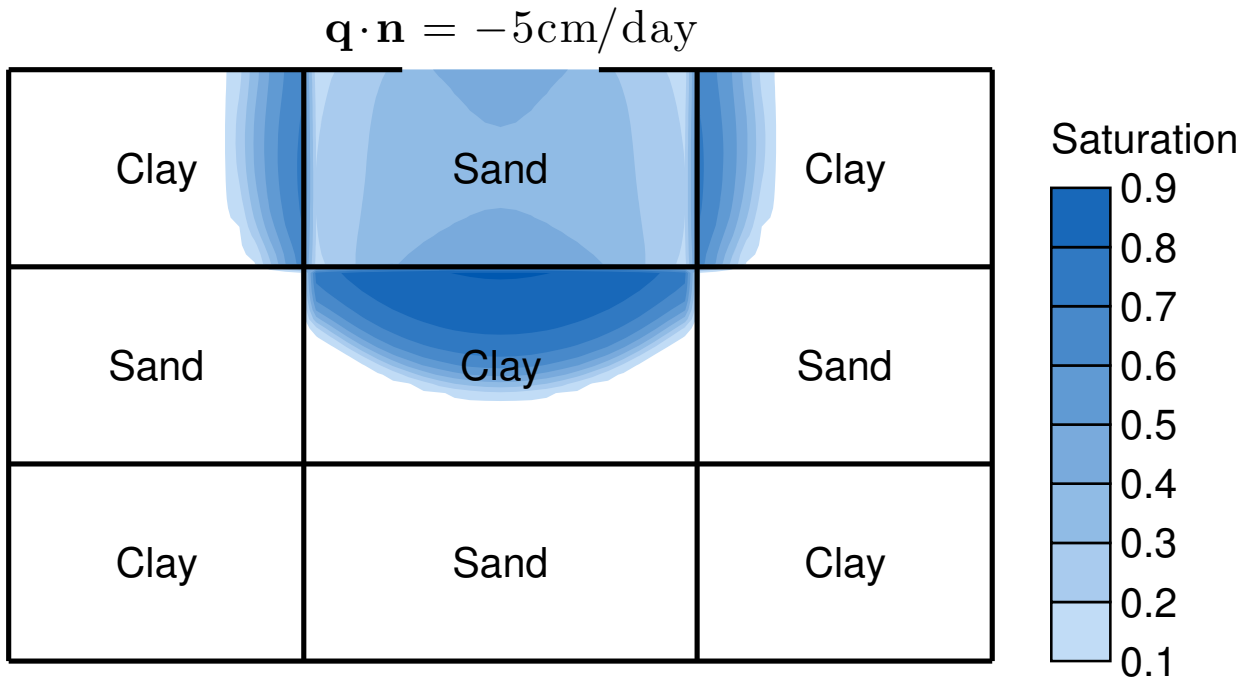
Test Problem: Water infiltration



Test Problem: Water infiltration



Test Problem: Water infiltration



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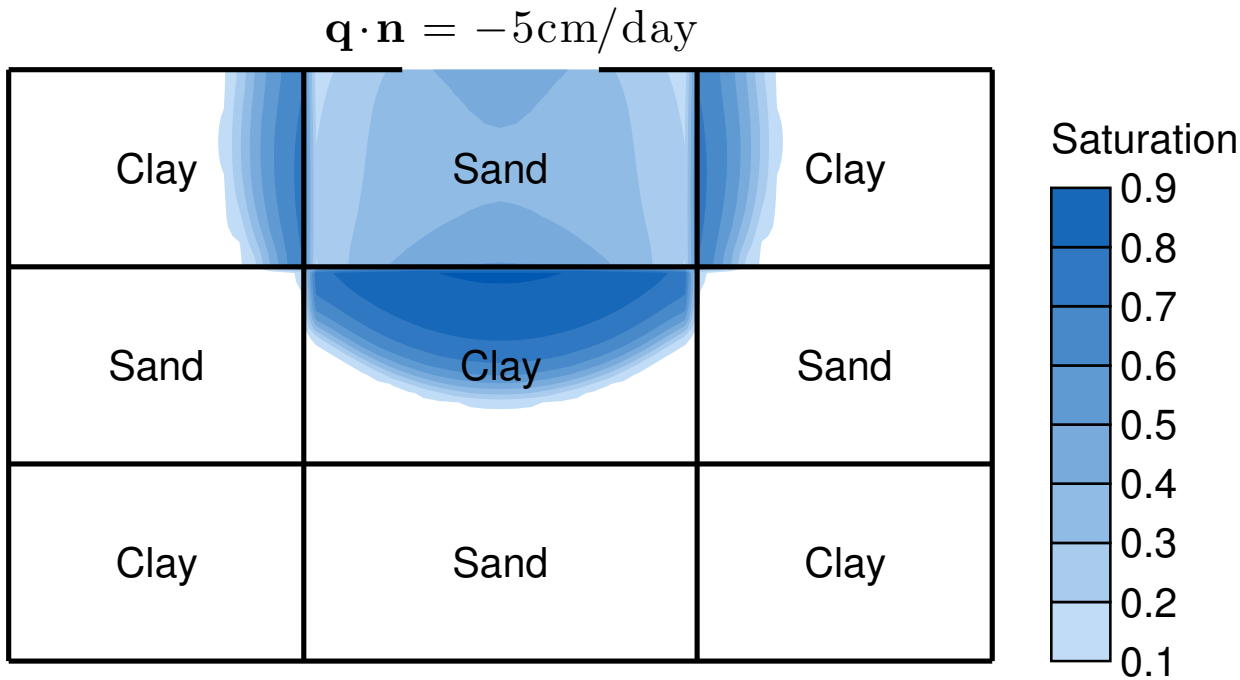

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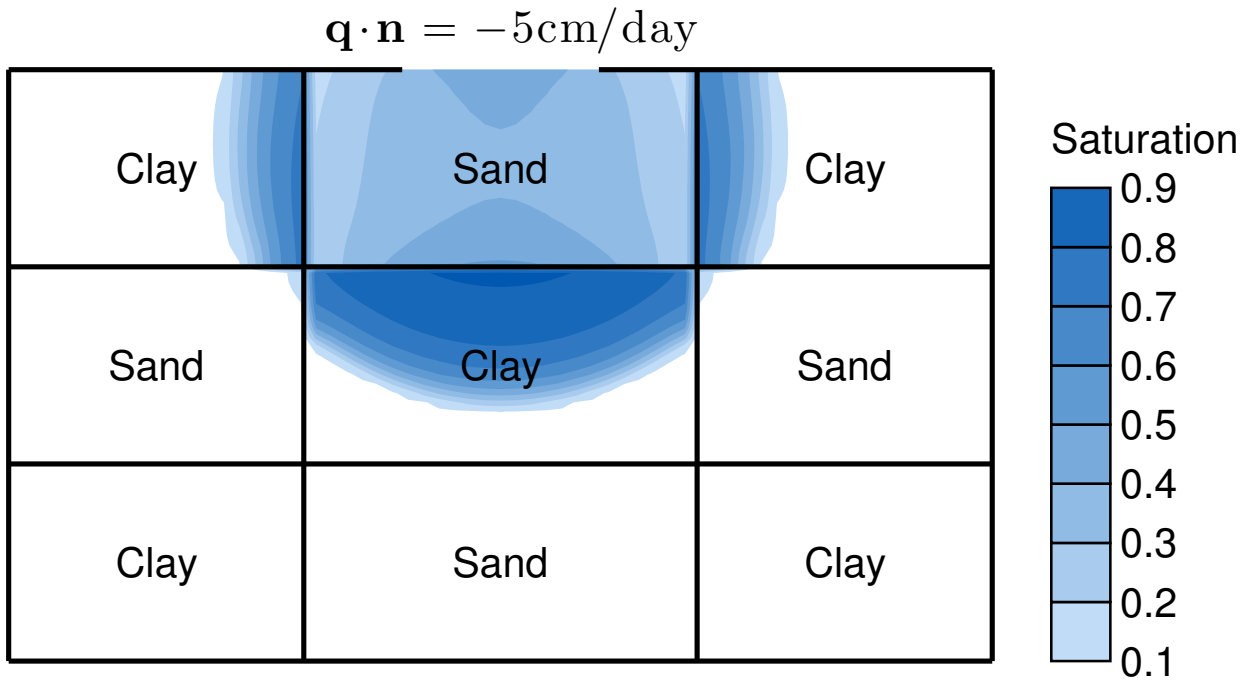
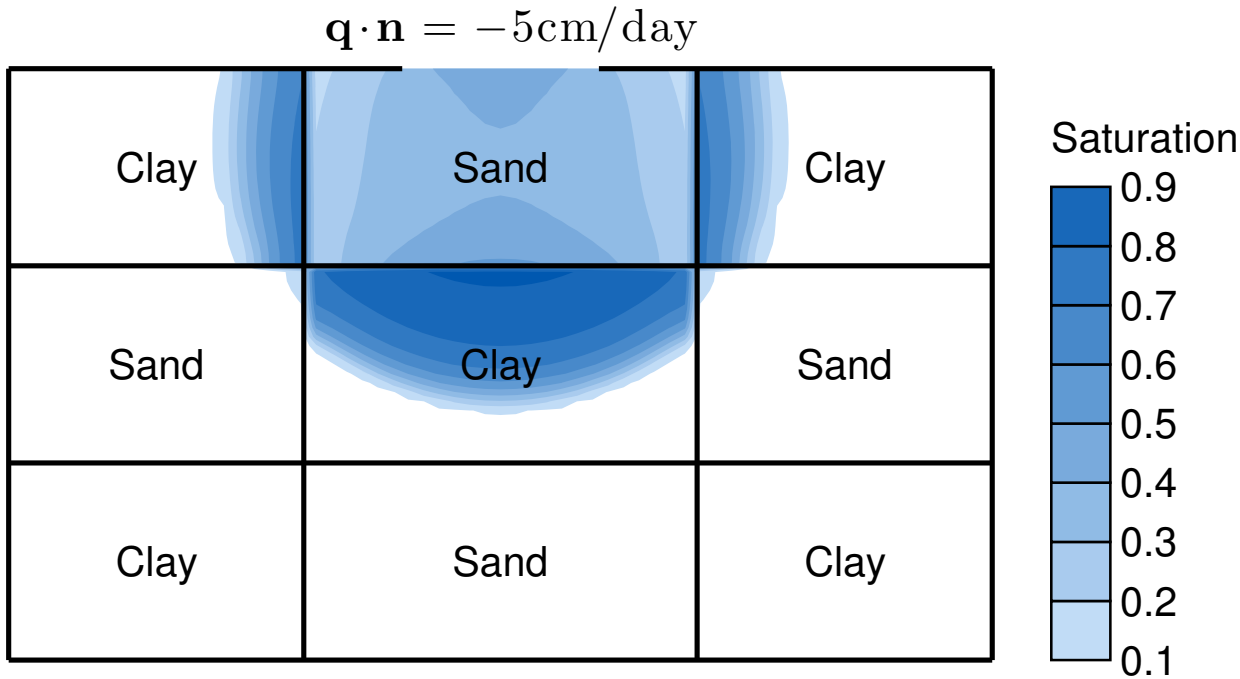
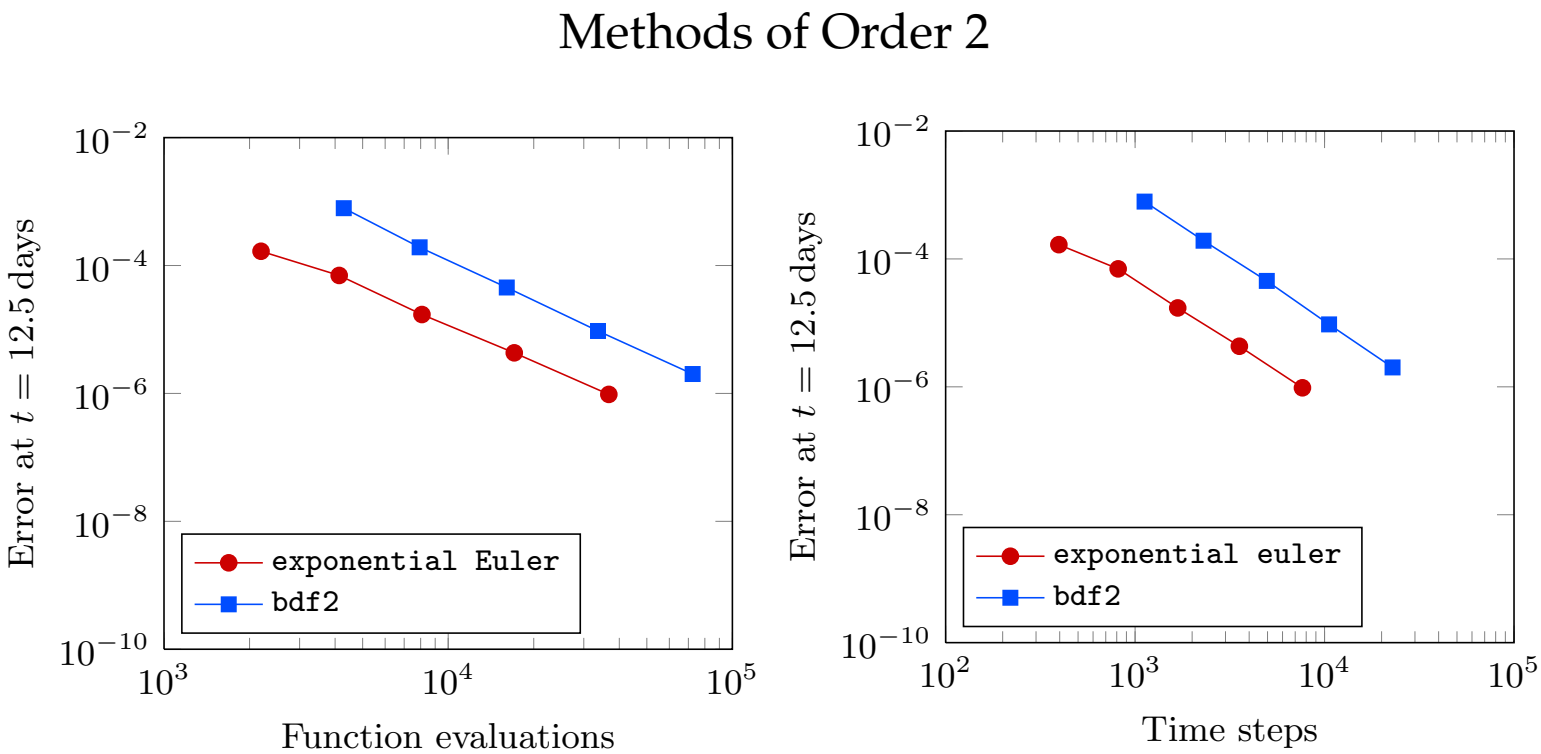


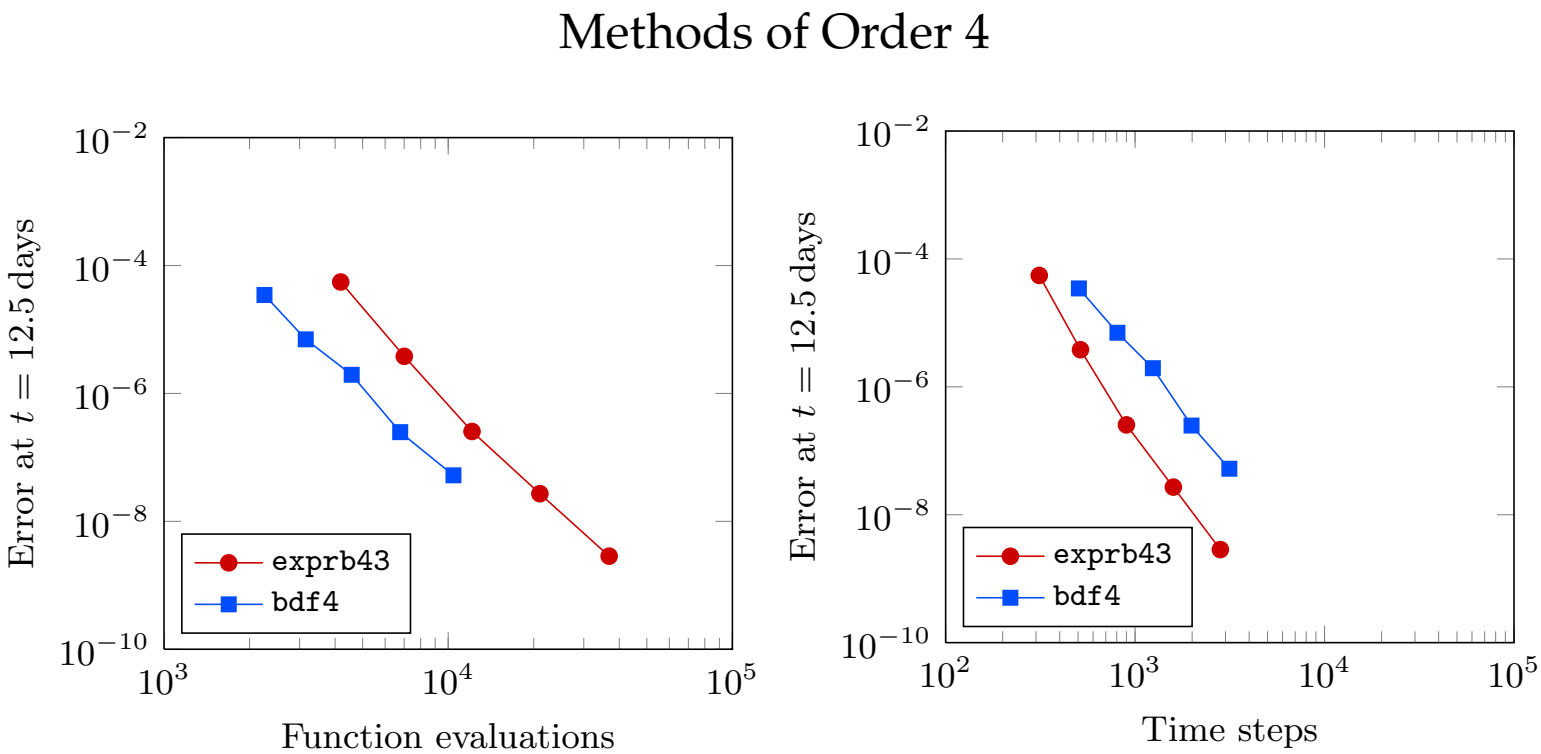
Test Problem: Water infiltration



Test Case: Unsaturated water flow



Test Case: Unsaturated water flow



Summary and Conclusions

Pros and Cons of exponential integrators:

- ✓ Krylov subspace methods converge rapidly
- ✓ Implemented efficiently without preconditioning $\Rightarrow$ Jacobian free
- ✓ Good stability properties $\Rightarrow$ Large time steps
- ✗ Spatial discretisation must be expressed in the form of

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{g}(\boldsymbol{u}), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0$$

- ✗ Performance declines in terms of function evaluations as we move up in order

Open Problem: Higher order methods require multiple matrix-function vector products with the same Jacobian matrix. How to recycle Krylov subspace information from one subspace to the next?

Overview of Presentation

- Modelling of unsaturated water flow
- Time integration methods for unsaturated water flow: Exponential integrators
- Two-scale numerical modelling of unsaturated water flow in soils exhibiting small-scale heterogeneities

Problem Statement

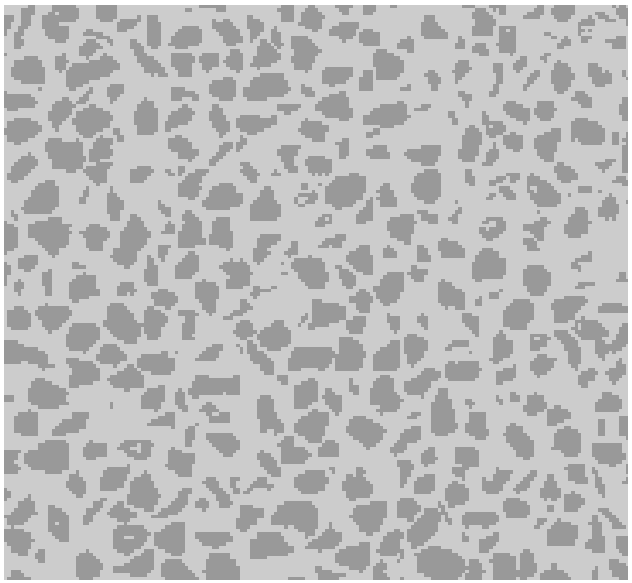
- Richards' equation in two-dimensions

$$\frac{\partial}{\partial t}\theta(h) + \nabla \cdot [-K(h)\nabla (h + x_2)] = 0$$

- Domain comprised of two sub-domains Ω_a (connected) and Ω_b (disconnected) with different hydraulic properties:

$$\theta(h) = \begin{cases} \theta_a(h) & \text{in } \Omega_a \\ \theta_b(h) & \text{in } \Omega_b \end{cases}$$

$$K(h) = \begin{cases} K_a(h) & \text{in } \Omega_a \\ K_b(h) & \text{in } \Omega_b \end{cases}$$



Heterogeneous domain

Ω_a Ω_b

- Computational cost of direct numerical simulation is prohibitively expensive when the domain exhibits small-scale heterogeneity.

Two-scale Model for unsaturated flow

- Macroscopic equation ($x \in \Omega$)

$$\frac{\partial}{\partial t}\theta_{\text{eff}}(h_a) + \nabla_x \cdot [-\boldsymbol{K}_{\text{eff}}(h_a)\nabla_x (h_a + x_2)] = Q$$

- Microscopic equation ($y \in \Omega_{x,b}$)

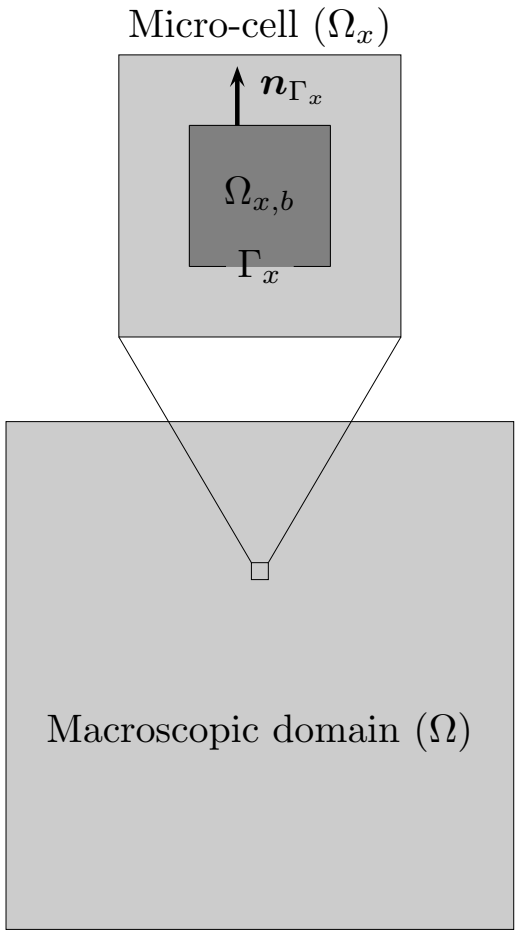
$$\frac{\partial}{\partial t}\theta_b(h_b) + \nabla_y \cdot [-K_b(h_b)\nabla_y (h_b + y_2)] = 0$$

- Microscopic boundary condition ($y \in \Gamma_x$)

$$h_a = h_b$$

- Source term

$$Q = \frac{1}{|\Omega_x|}\int_{\Gamma_x} -K_b(h)\nabla_y (h_b + y_2) \cdot \boldsymbol{n}_{\Gamma_x}\, ds$$



See, e.g., Szymkiewicz and Lewandowska (2006).

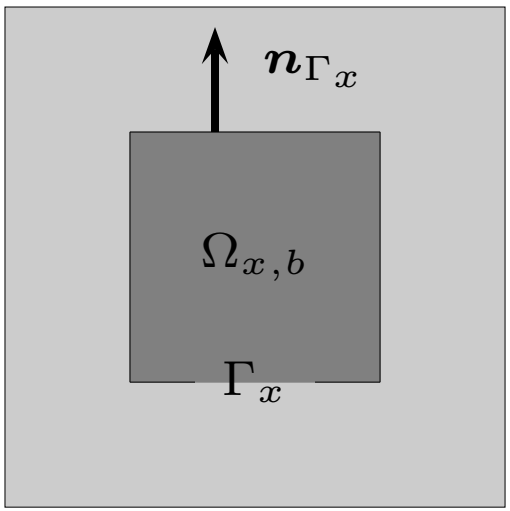
Effective hydraulic conductivity

- $(\boldsymbol{K}_{\text{eff}})_{*,j}$ is a matrix with jth column equal to

$$(\boldsymbol{K}_{\text{eff}})_{*,j} = \frac{1}{|\Omega_x|} \int_{\Omega_{x,a}} K_a(h_a) \nabla_y (u + y_2) \, dA$$

where u is a solution of the periodic cell-problem:

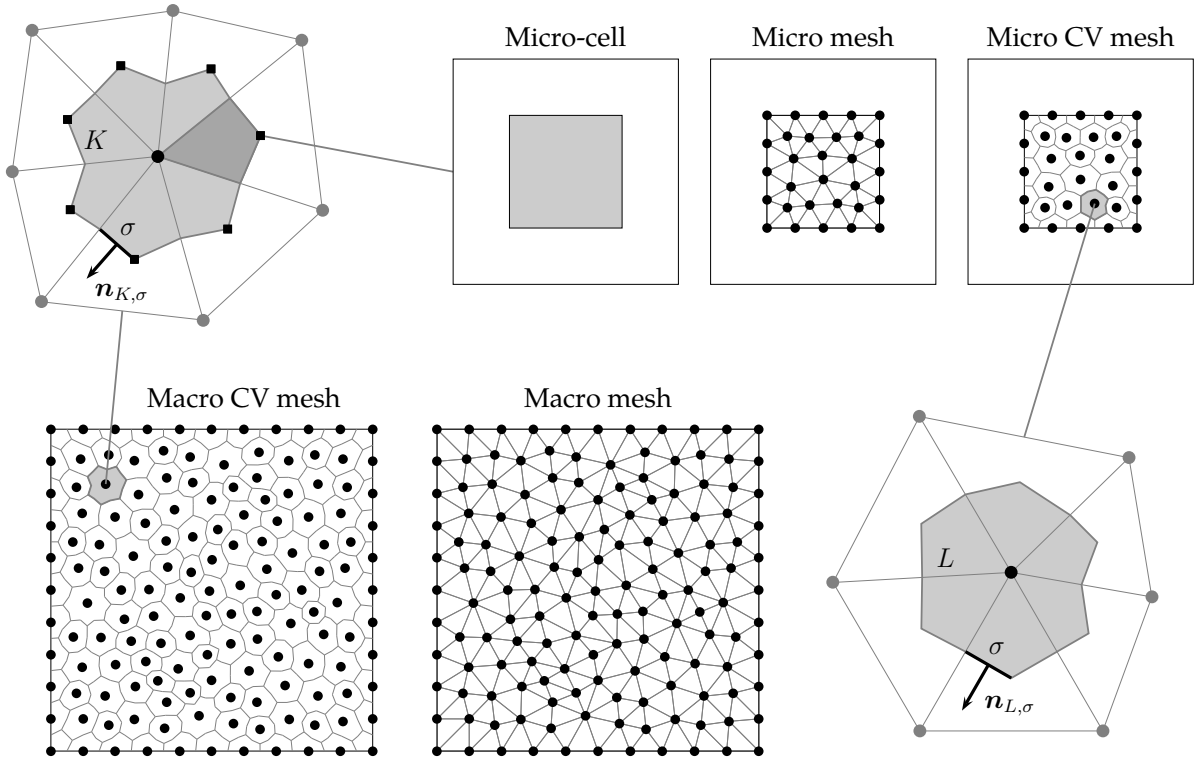
$$\begin{cases} \nabla_y \cdot (K_a(h_a) \nabla_y (u + y_2)) = 0 & y \in \Omega_{x,a} \\ K_a(h_a) \nabla_y (u + y_2) \cdot \boldsymbol{n}_{\Gamma_x} = 0 & y \in \Gamma_x \\ u \text{ is } \Omega_x\text{-periodic} \end{cases}$$



- Pre-processing step:
  (a) Compute entries of $\boldsymbol{K}_{\text{eff}}$ using a range of values of h_a
  (b) Use interpolation to evaluate $\boldsymbol{K}_{\text{eff}}$ at any value of h_a

Unstructured CV spatial discretisation

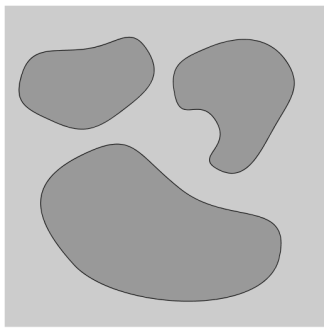
- Unstructured vertex-centered CV method [Carr and Turner (2014)]



- Existing numerical strategies (e.g., Szymkiewicz and Lewandowska (2008)) only applicable to simple inclusion geometries

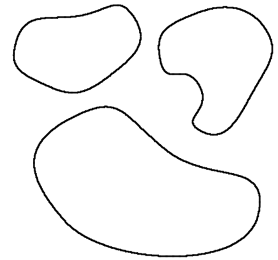
Meshing of micro-cells

Image-based meshing code built on GMSH [Guezaine and Remacle (2009)]

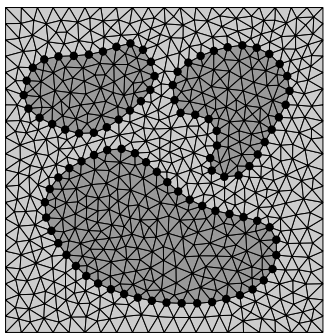


Bitmap image

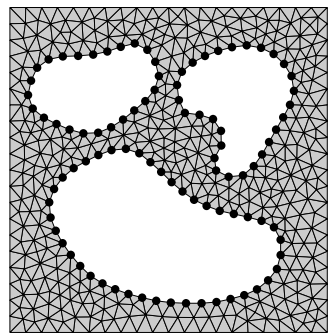
Binary image



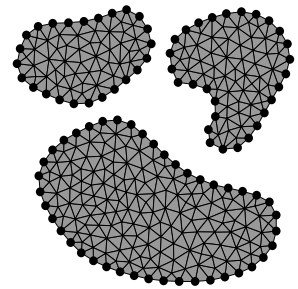
Edge detection



Full micro-cell



Ω_a only



Ω_b only

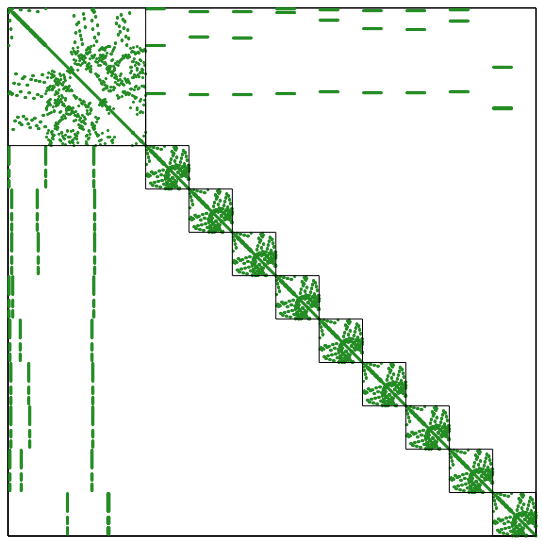
Spatially-discrete system of ODEs

- Spatial discretisation can be expressed in the form

$$\frac{d\boldsymbol{u}}{dt} = \boldsymbol{g}(\boldsymbol{u}), \quad \boldsymbol{u}(0) = \boldsymbol{u}_0 \in \mathbb{R}^N,$$

  where $N = \text{num\_macro\_nodes} + \text{num\_macro\_elements} \times \text{num\_micro\_nodes}$.

- Example sparsity structure of Jacobian matrix (zoomed in)



$$\boldsymbol{J}(\boldsymbol{u})\boldsymbol{v} \approx [\boldsymbol{g}(\boldsymbol{u} + \varepsilon\boldsymbol{v}) - \boldsymbol{g}(\boldsymbol{u})]/\varepsilon$$

- Time integration: Jacobian-free exponential Euler method [Carr et al. (2011)]

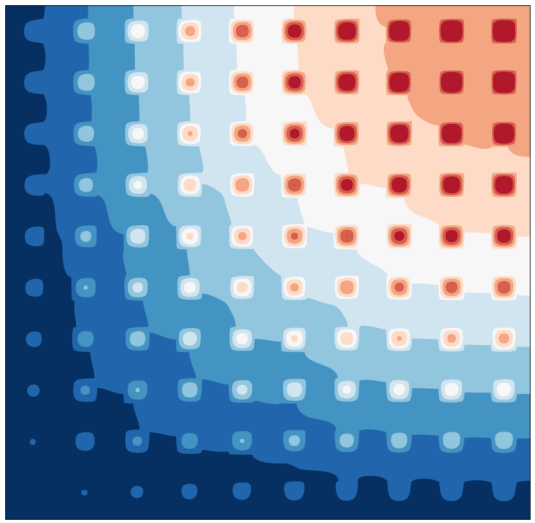
Code implementation details

- Code developed in C++
- Linear algebra operations performed using BLAS and LAPACK libraries:
  (a) Intel MKL (HPC Platform)
  (b) Accelerate framework (Macbook Pro)
- ODE right-hand side function $\boldsymbol{g}(\boldsymbol{u})$ implemented in parallel using OpenMP

```
#include <omp.h>
omp_set_num_threads(NUM_THREADS);
#pragma omp parallel for
for (k=0; k<num_macro_elements; k++)
{
  ...
}
```

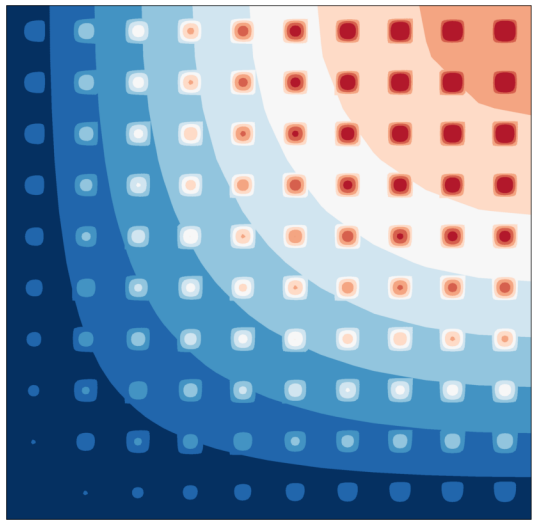
- Code accommodates both triangular and quad elements at both scales

Test Case: Diffusion equation



Full Model
(15 mins)

Macroscopic Model
(1 sec)

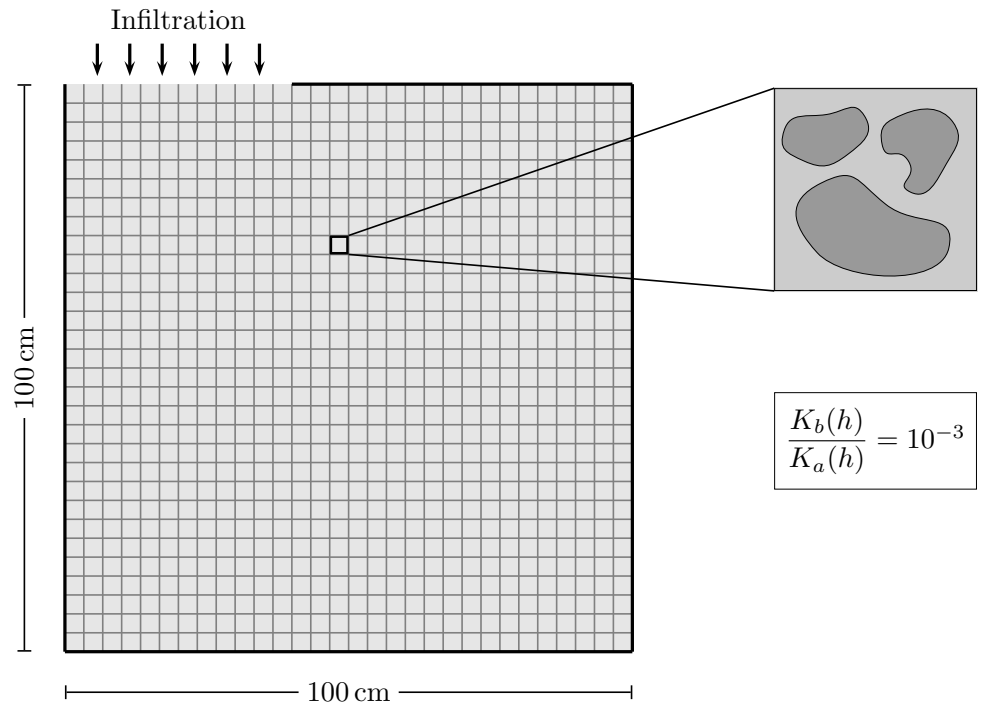


Two-scale Model
(20 secs)

Mesh details:

- Full Model: 400×400 grid, $N = 160,801$
- Macroscopic Model: 20×20 grid, $N = 441$
- Two-scale Model: Macro (20×20 grid), Micro (10×10 grid), $N = 48,841$

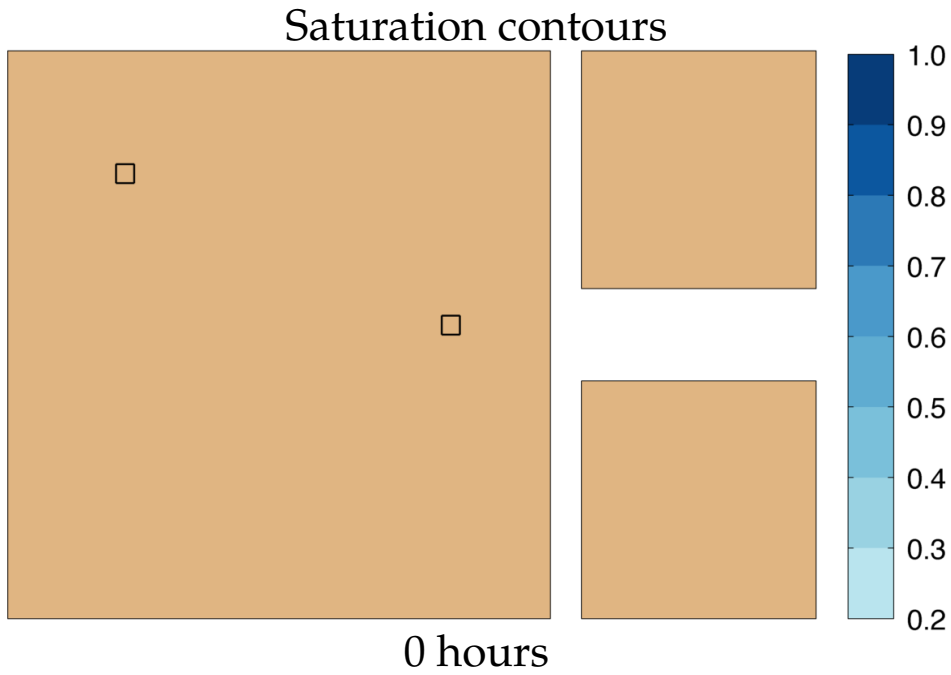
Simulation: Unsaturated water flow



Mesh details:

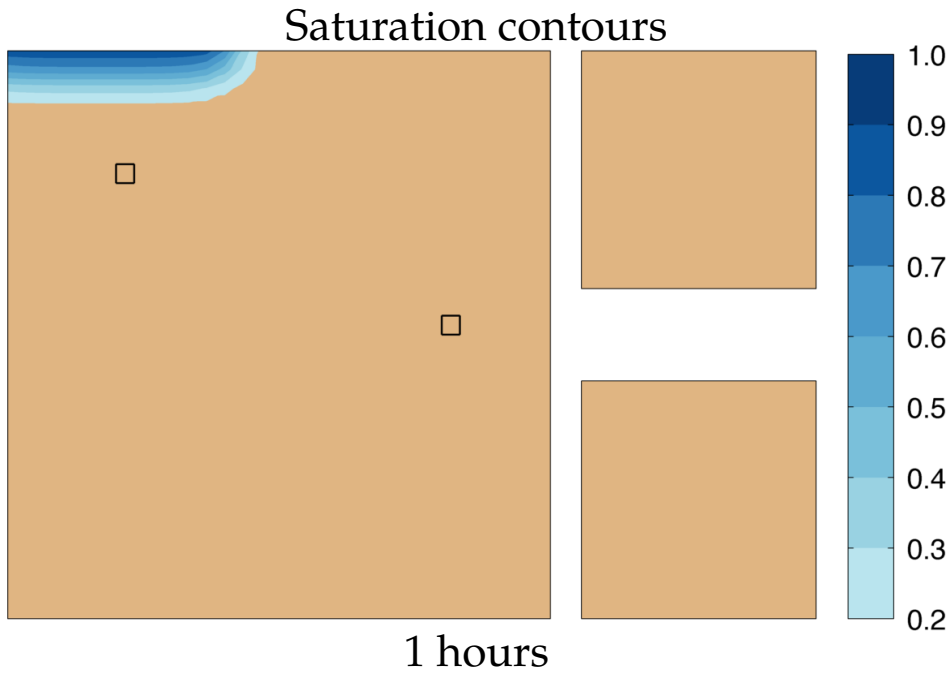
- Macroscopic mesh (structured): 30×30 grid, 961 nodes
- Microscopic mesh (unstructured): 532 elements, 316 nodes
- Total number of unknowns: $N = 961 + 900 \times 316 = 285,361$

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



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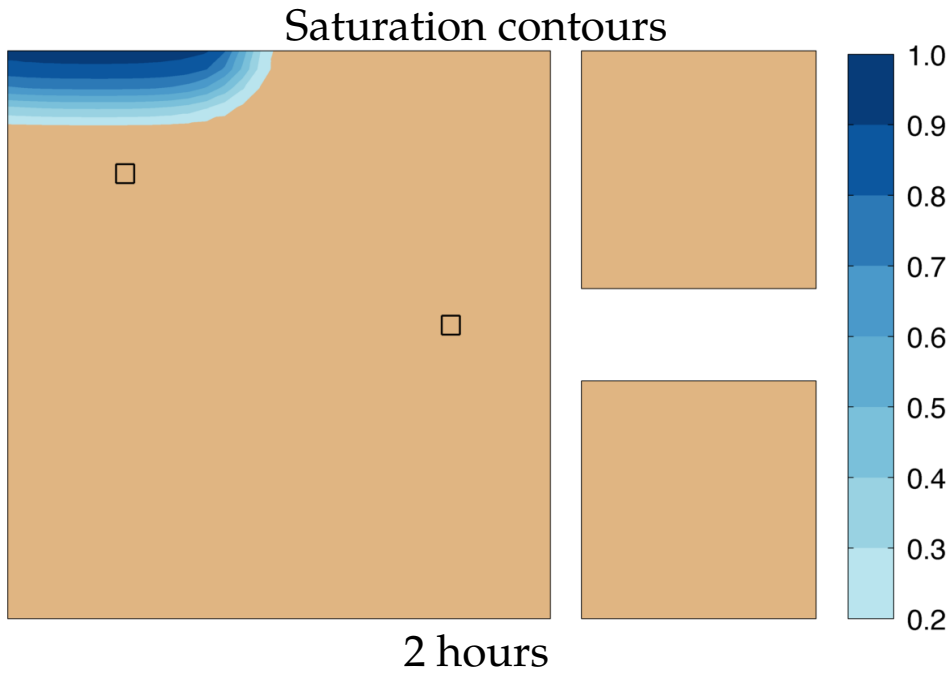
Simulation: Unsaturated water flow



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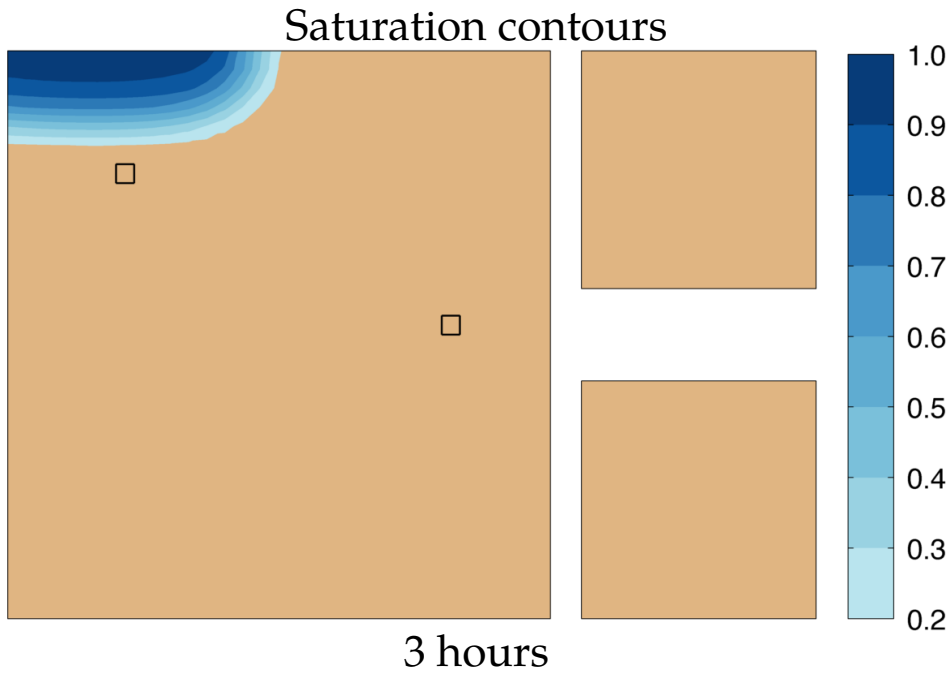
Simulation: Unsaturated water flow



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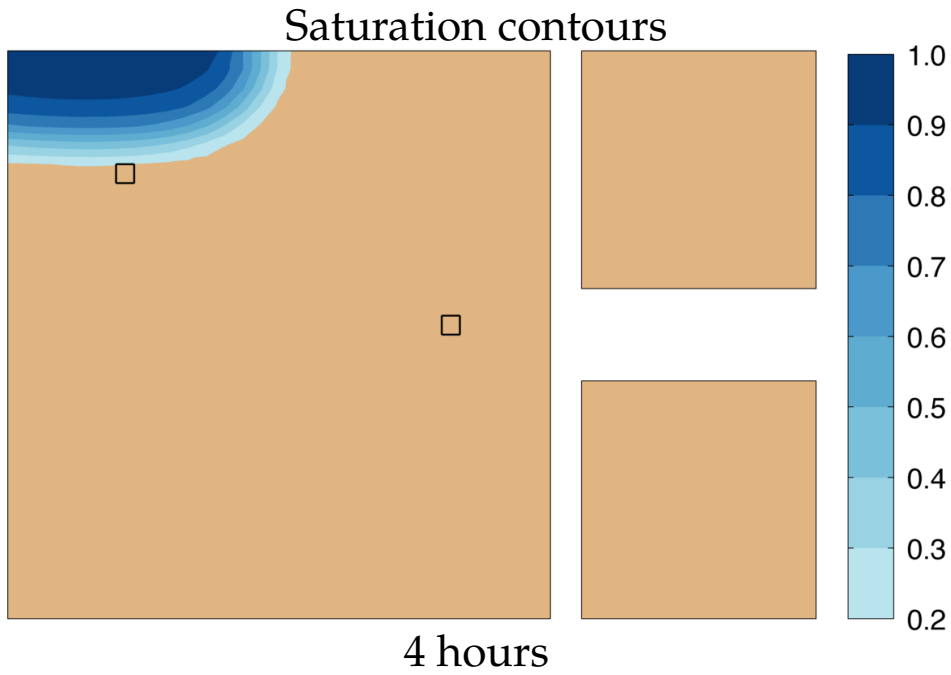

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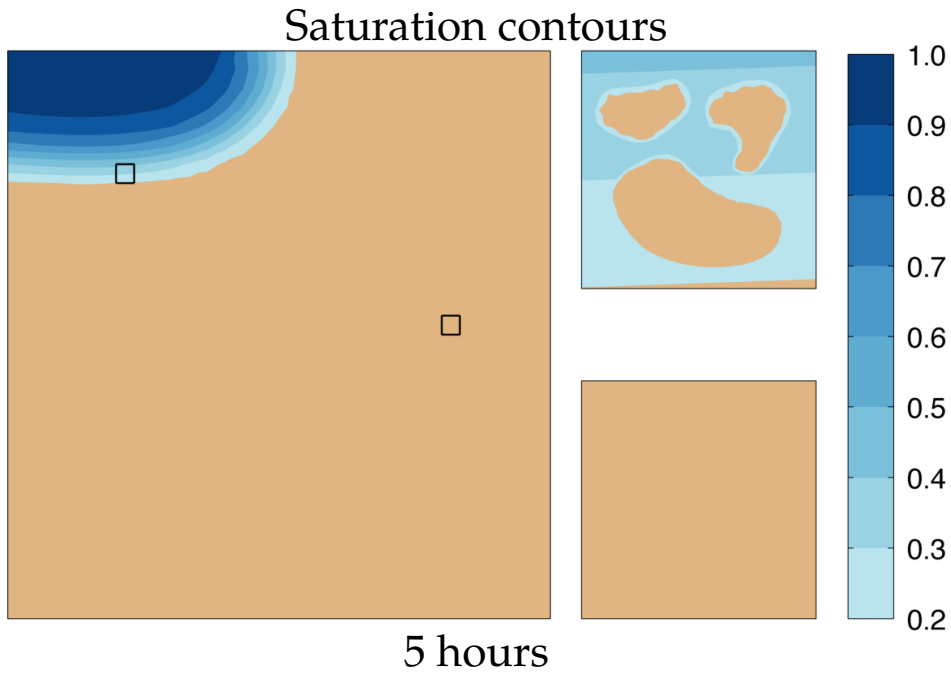

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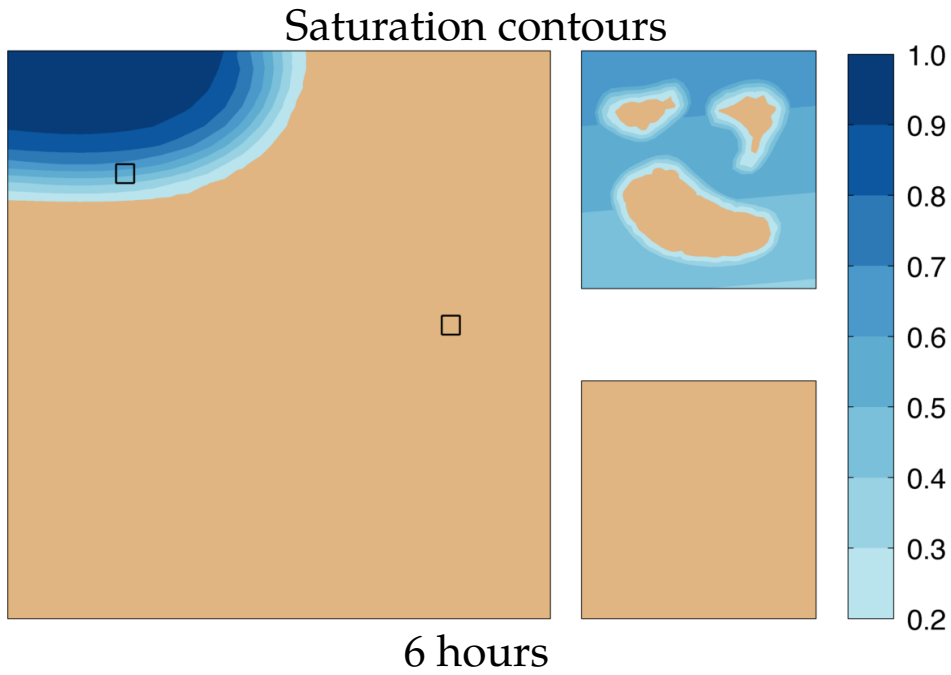
Simulation: Unsaturated water flow



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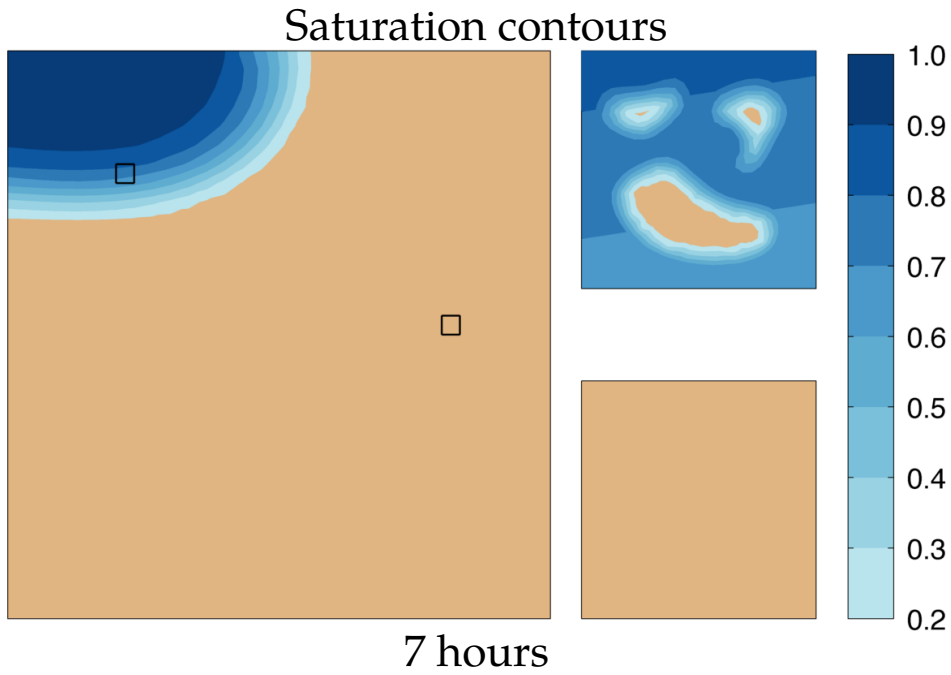
Simulation: Unsaturated water flow



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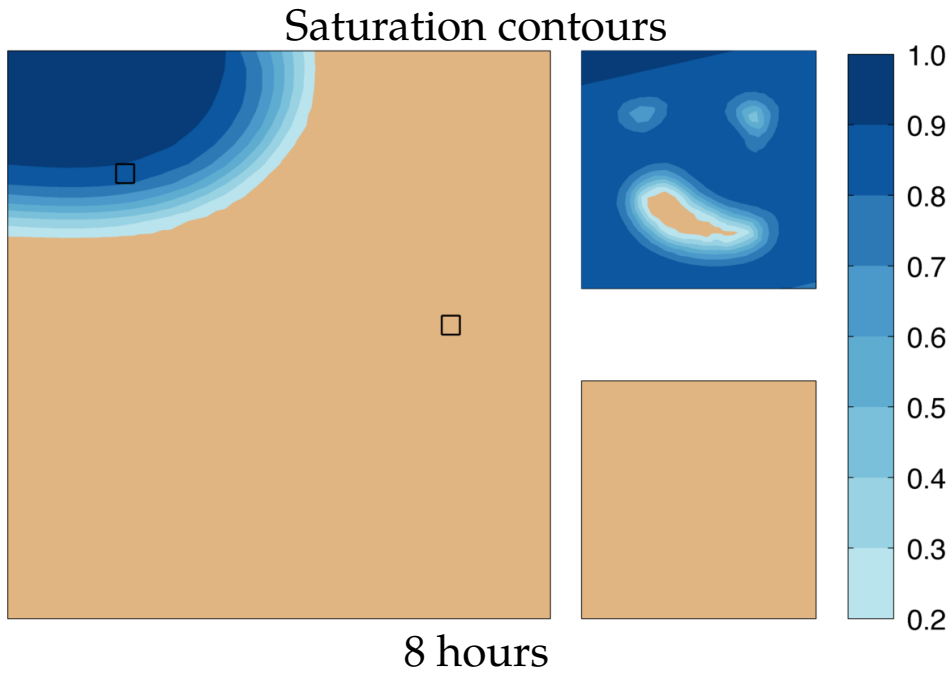
Simulation: Unsaturated water flow



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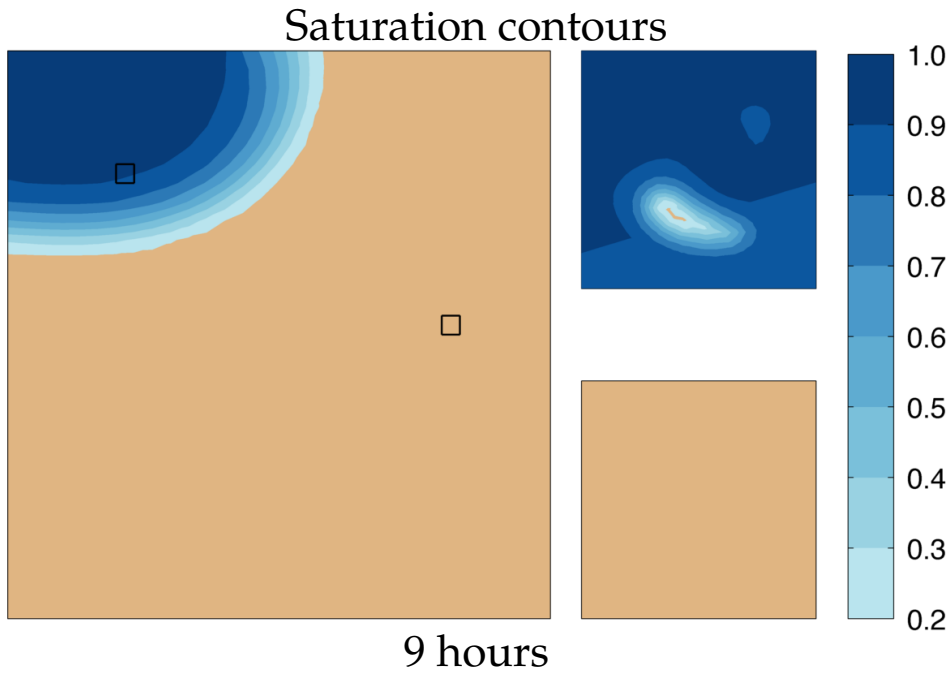

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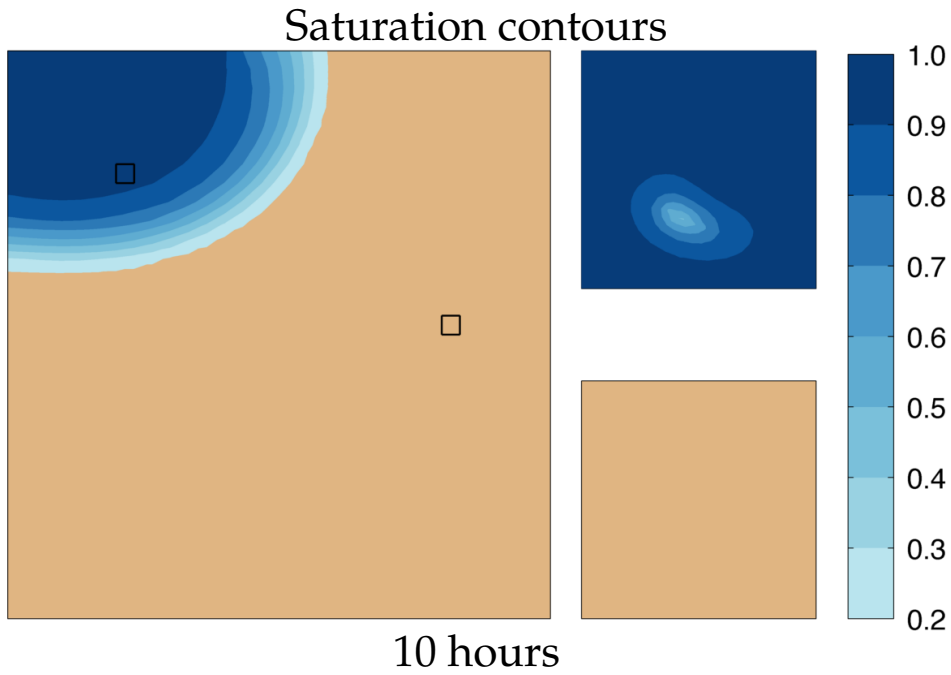
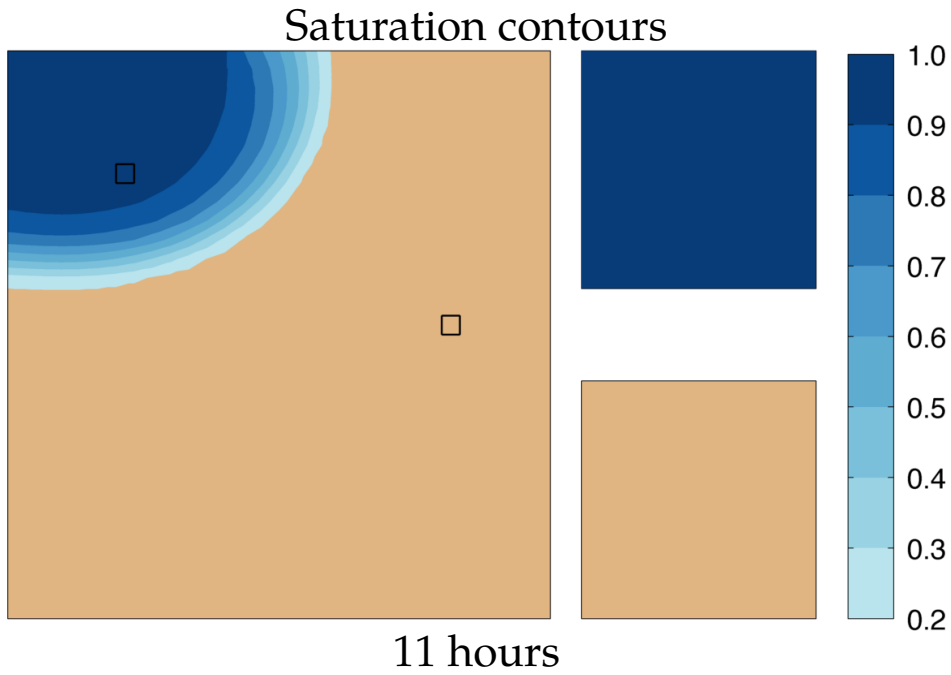
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



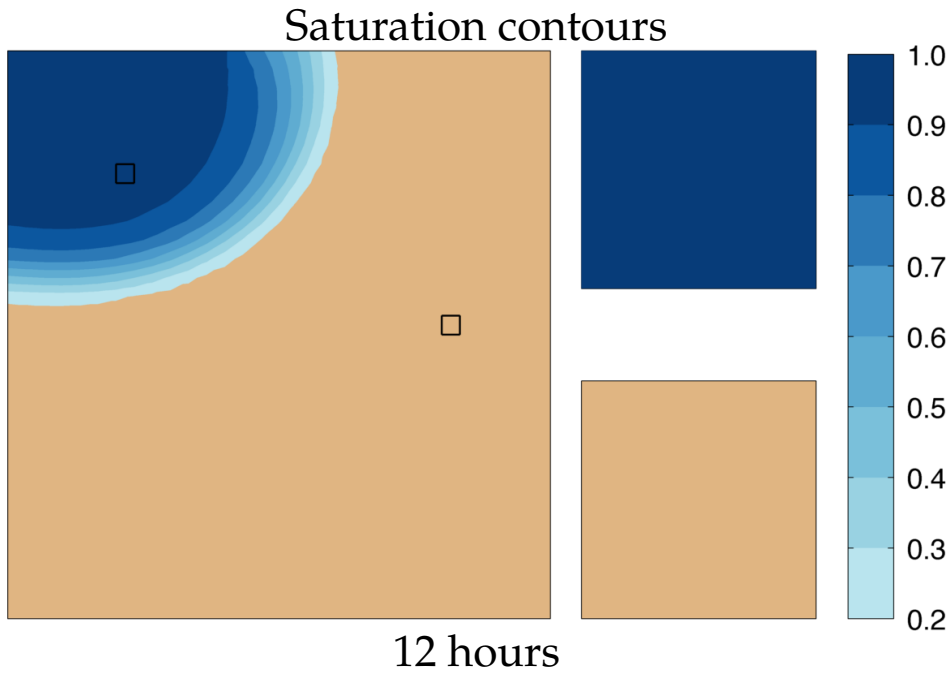
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



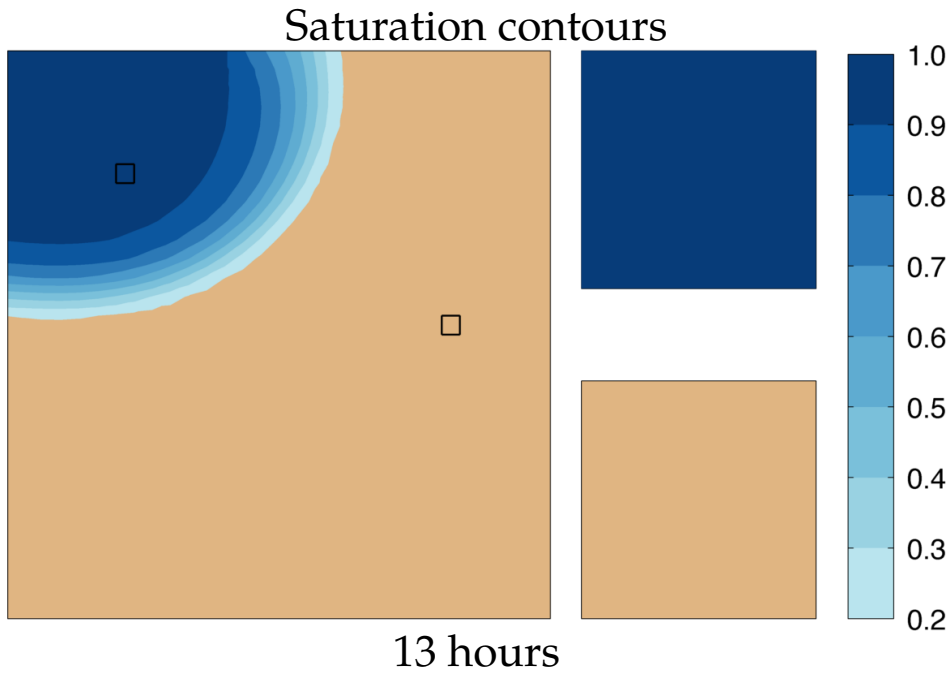
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



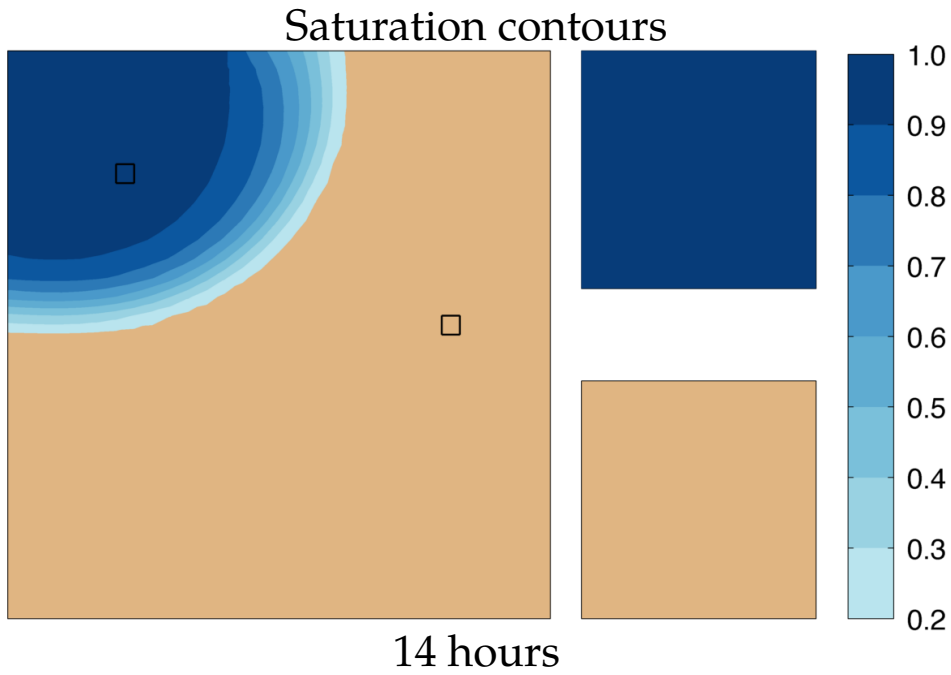
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



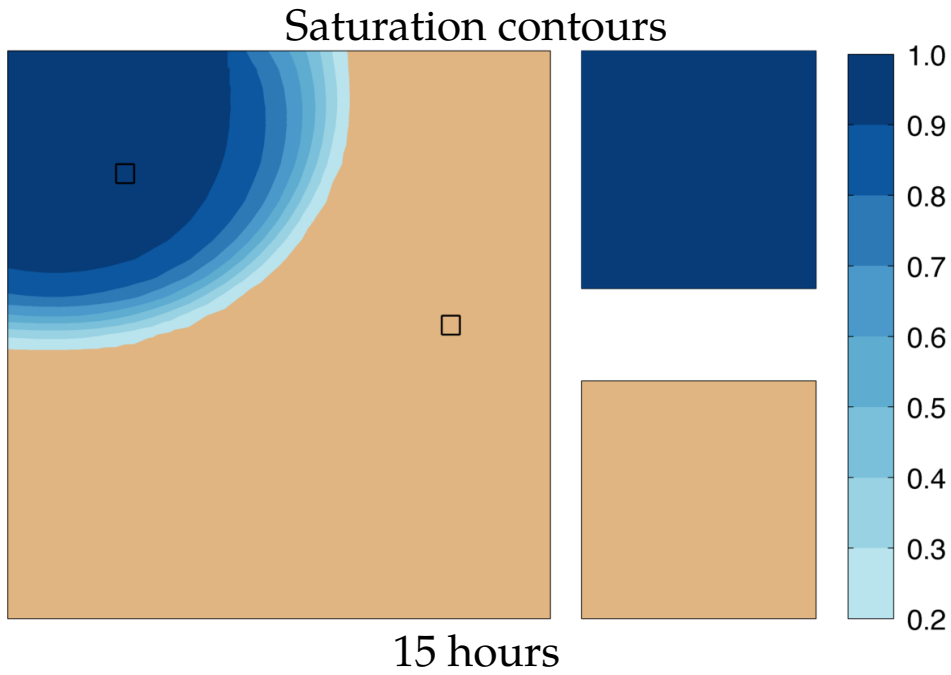
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



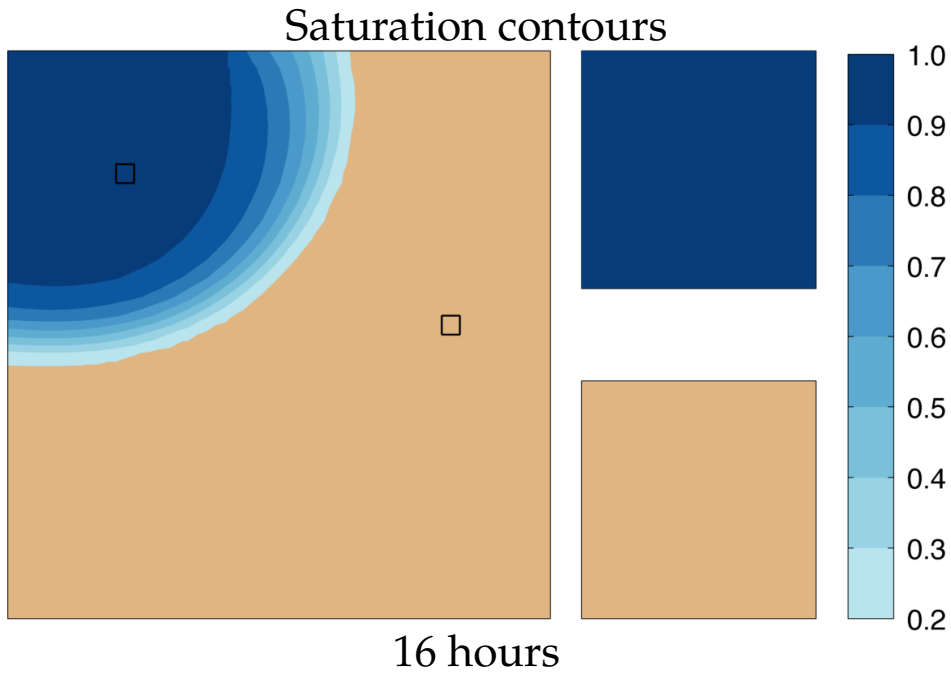
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



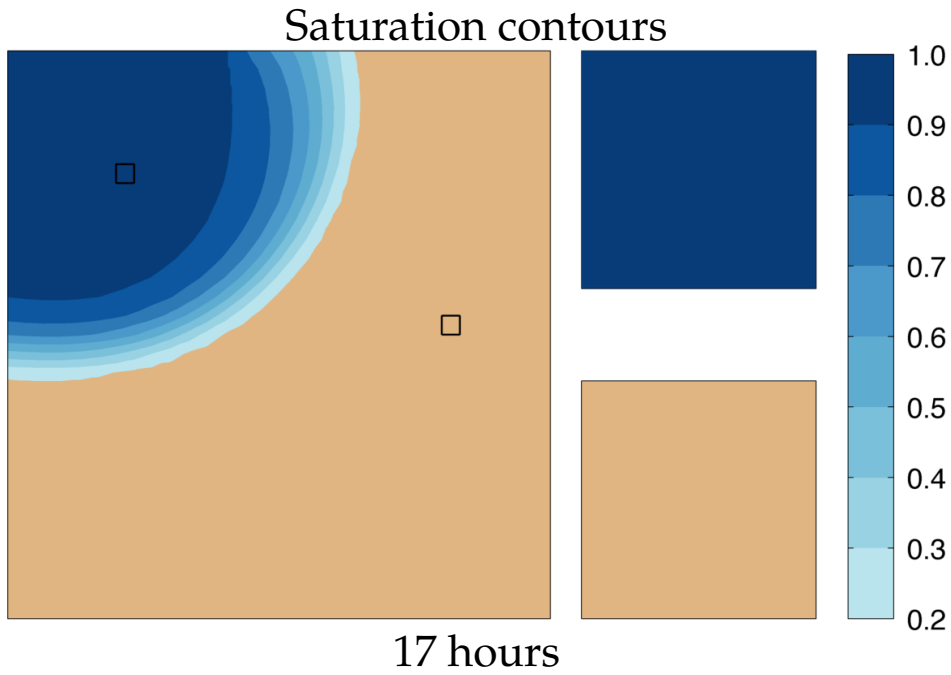
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



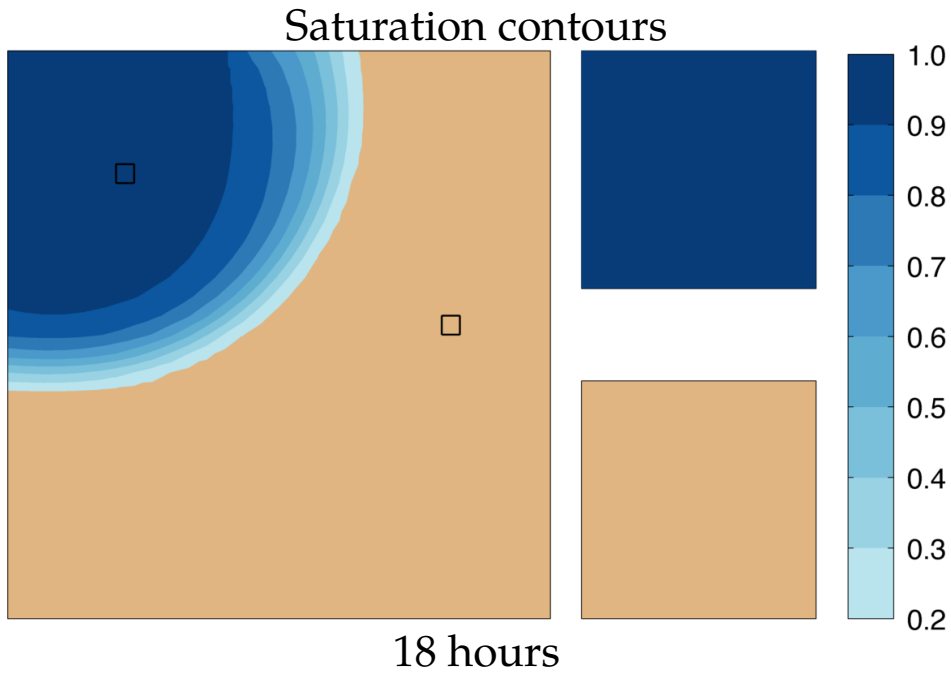
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



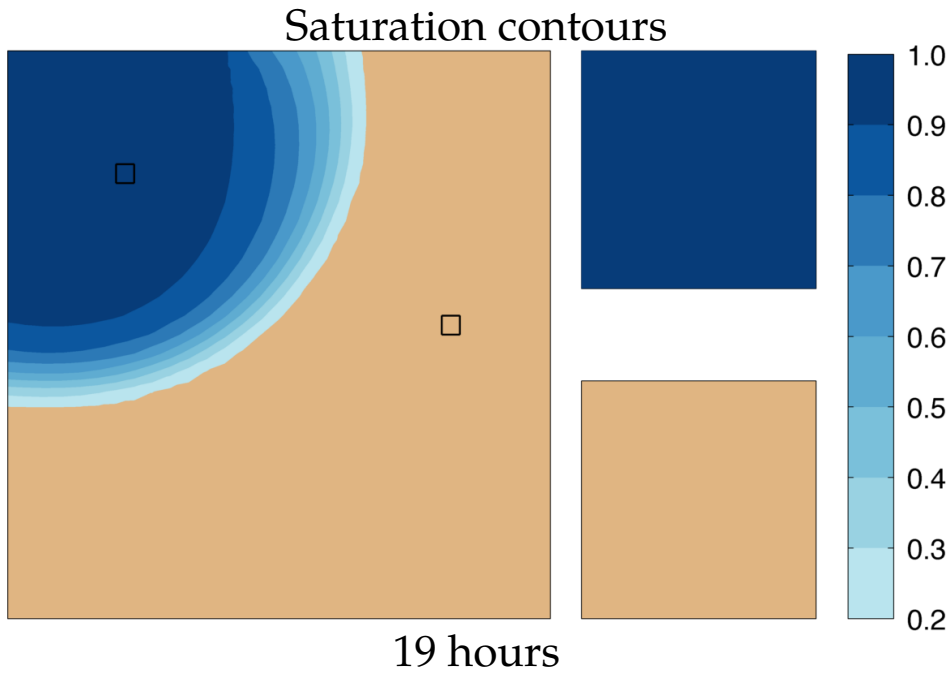
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



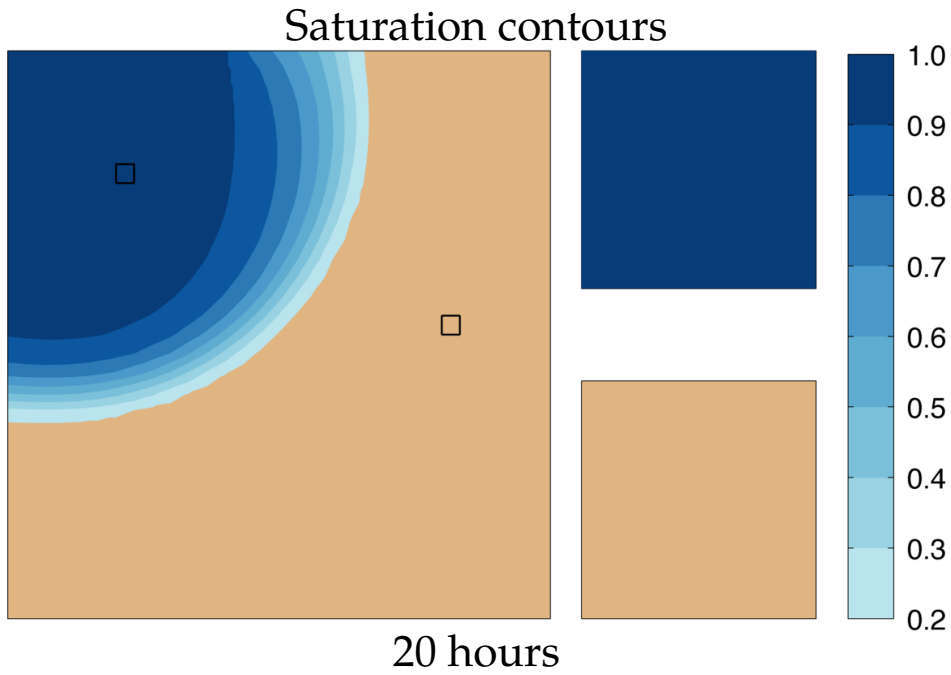
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



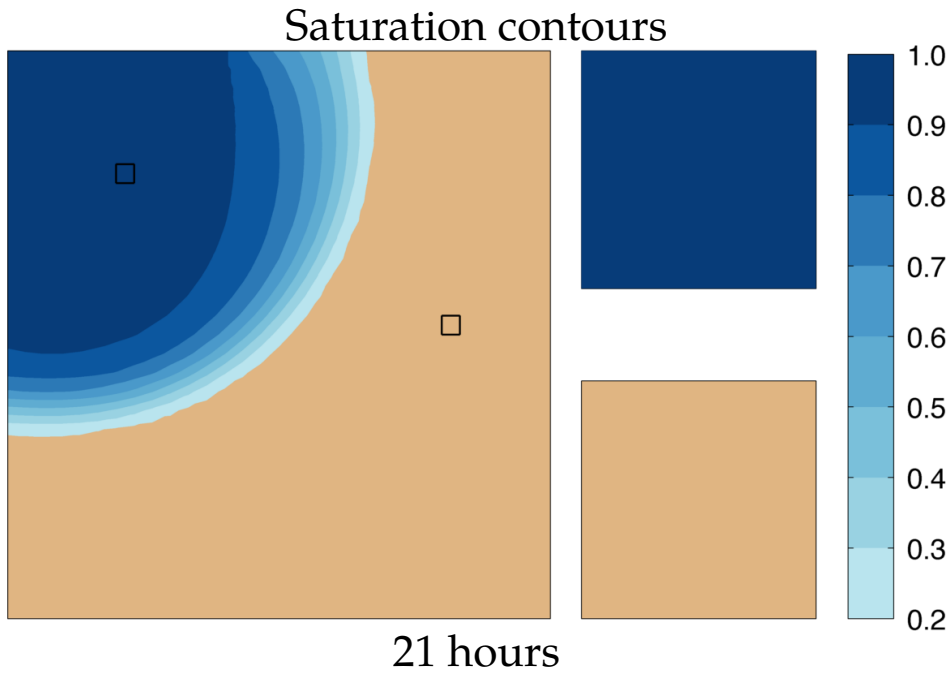
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



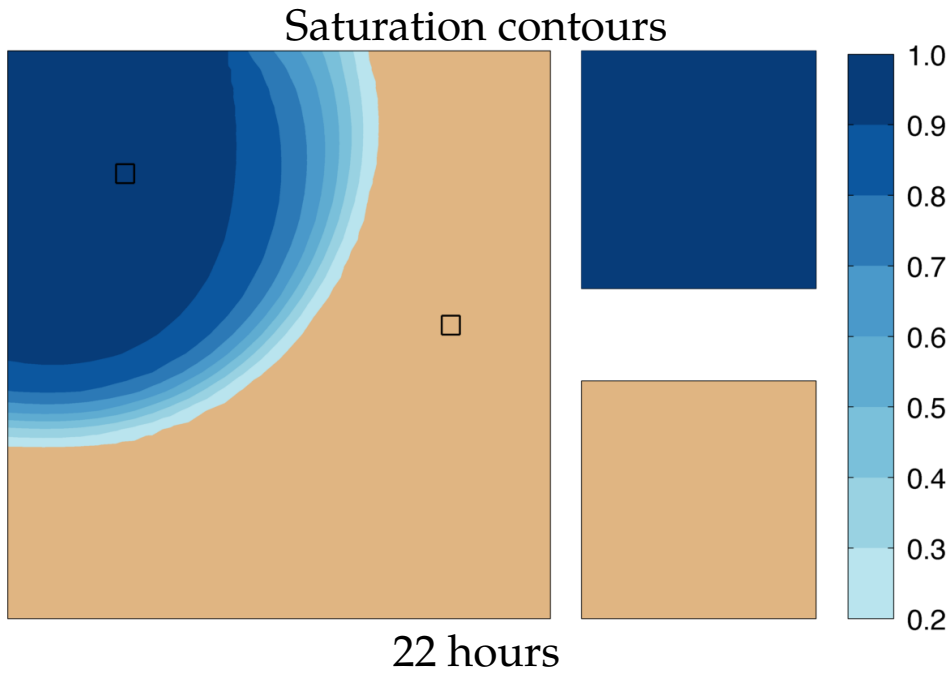
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



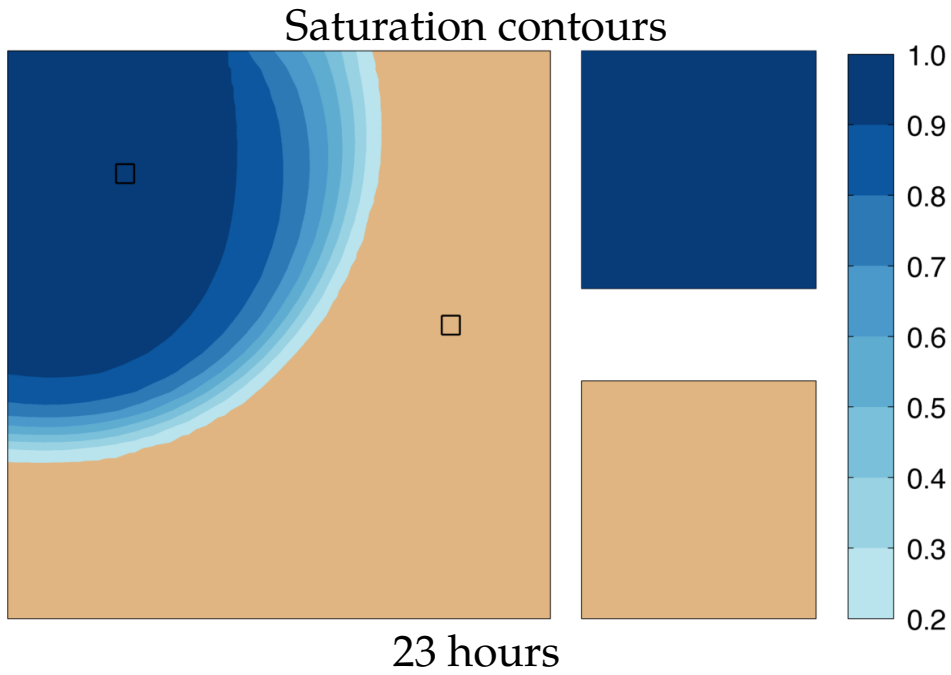
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



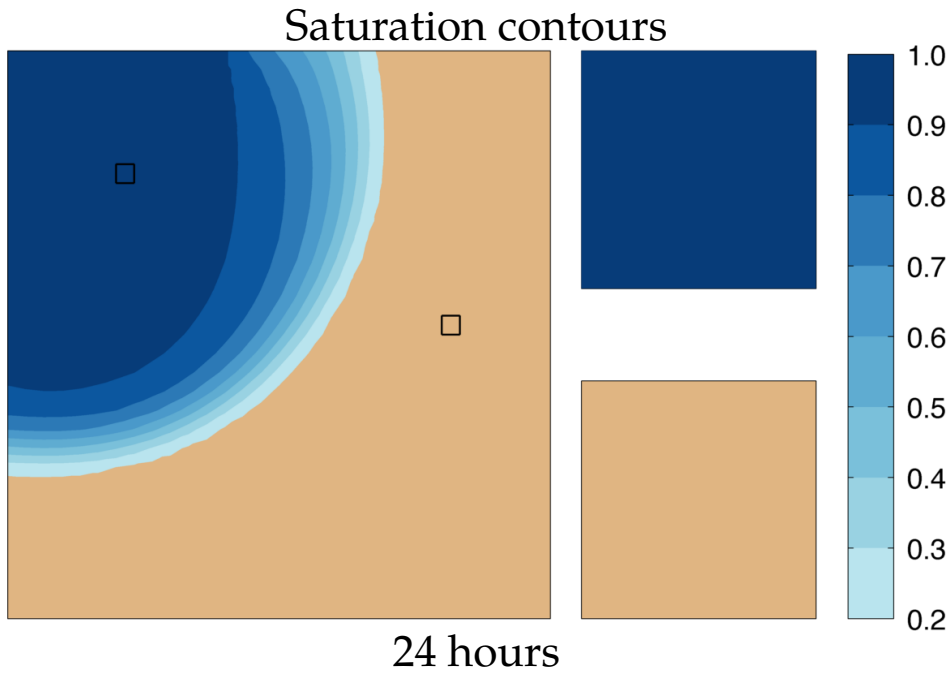
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



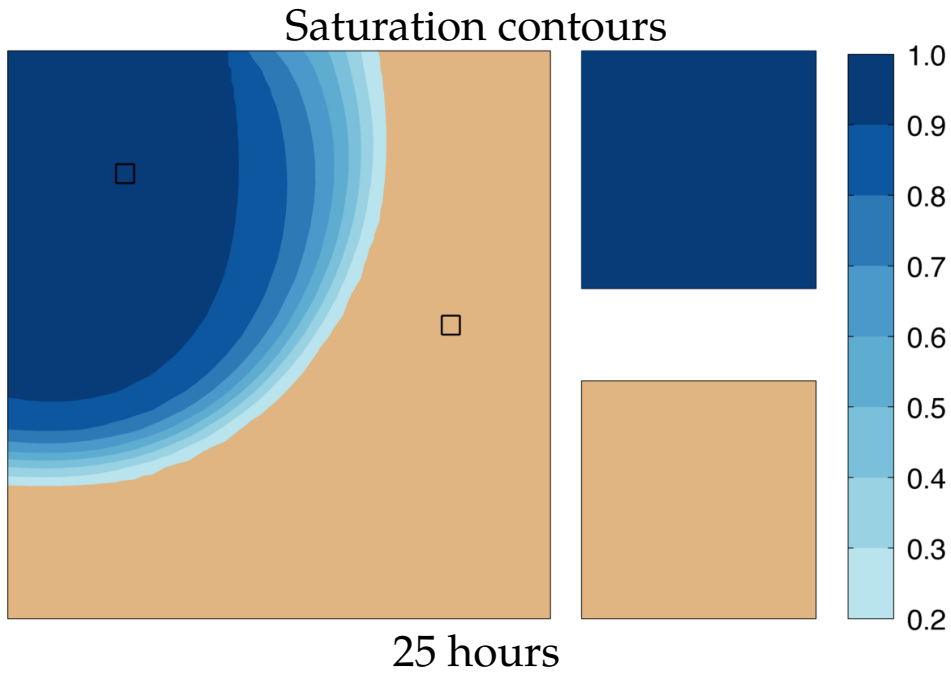
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



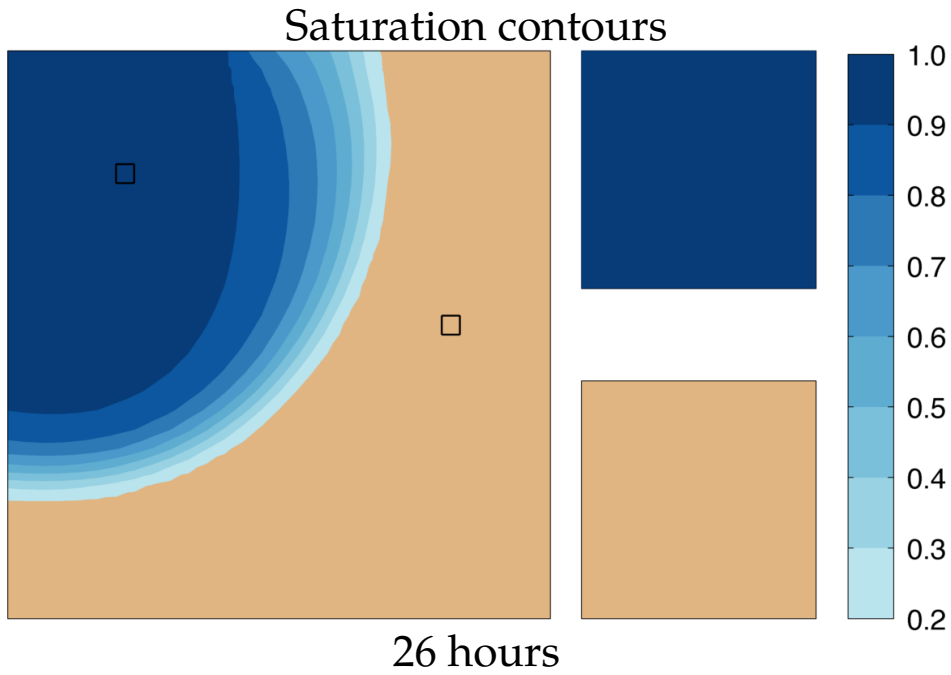
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



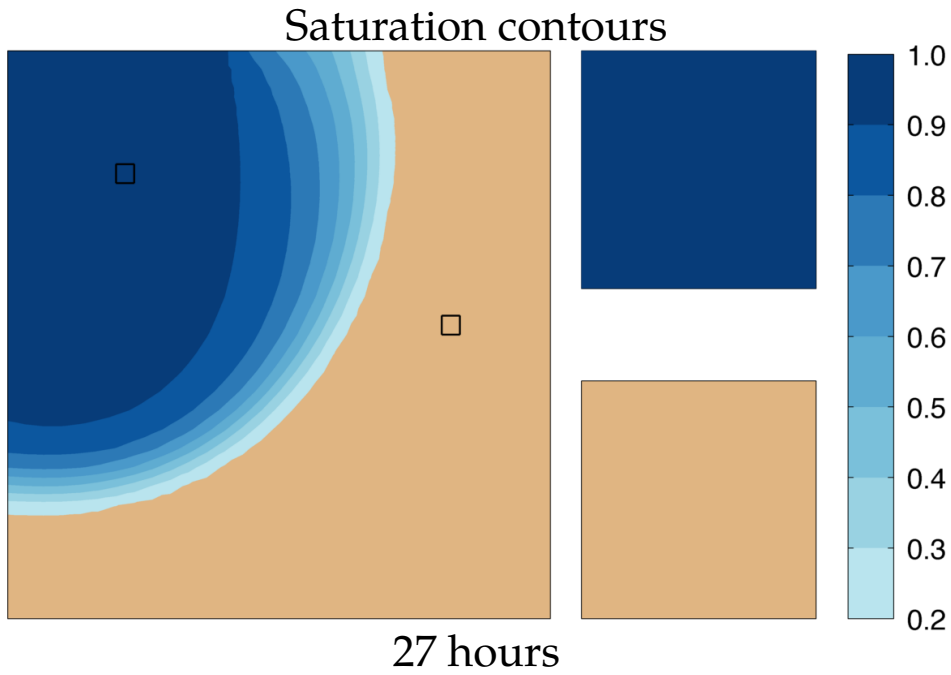
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



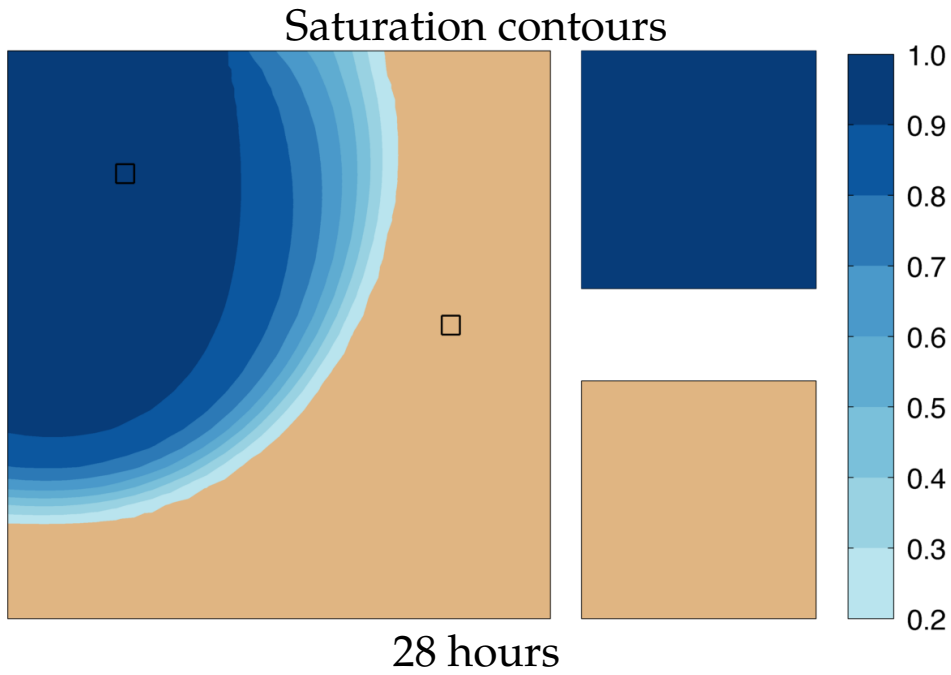
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



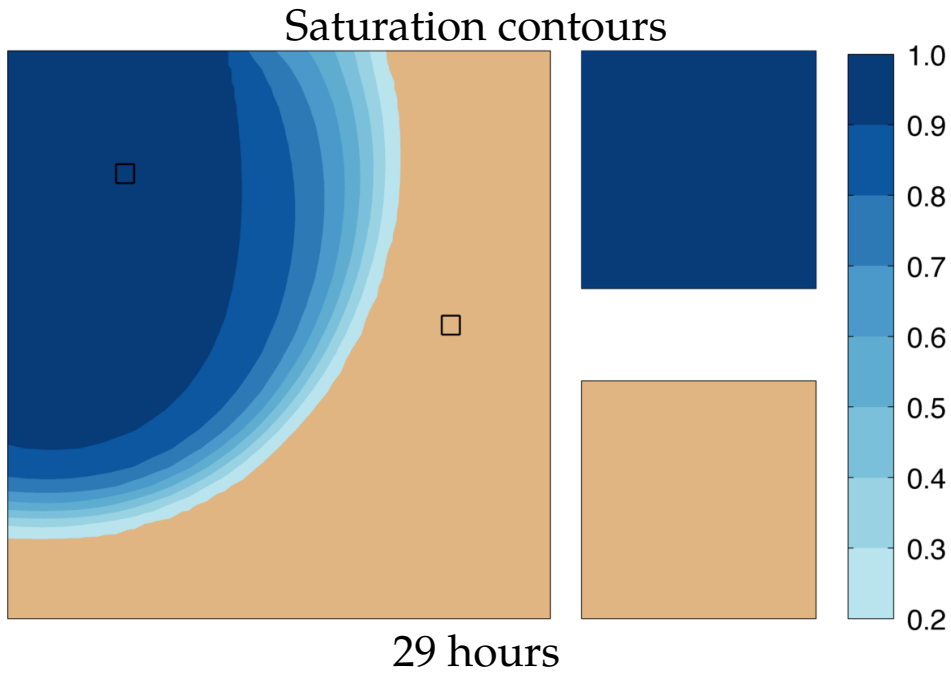
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



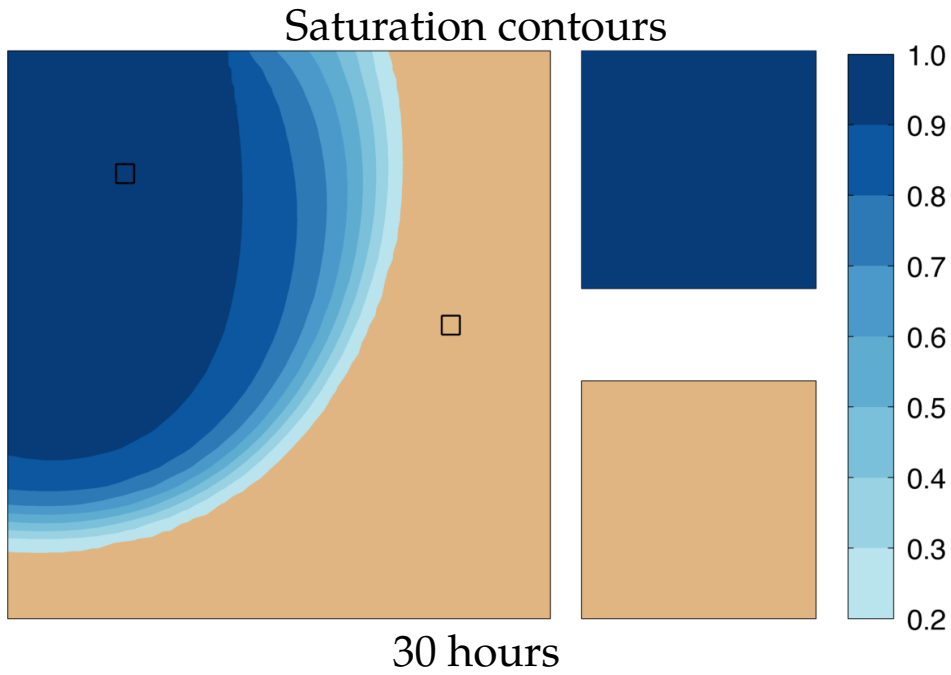
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



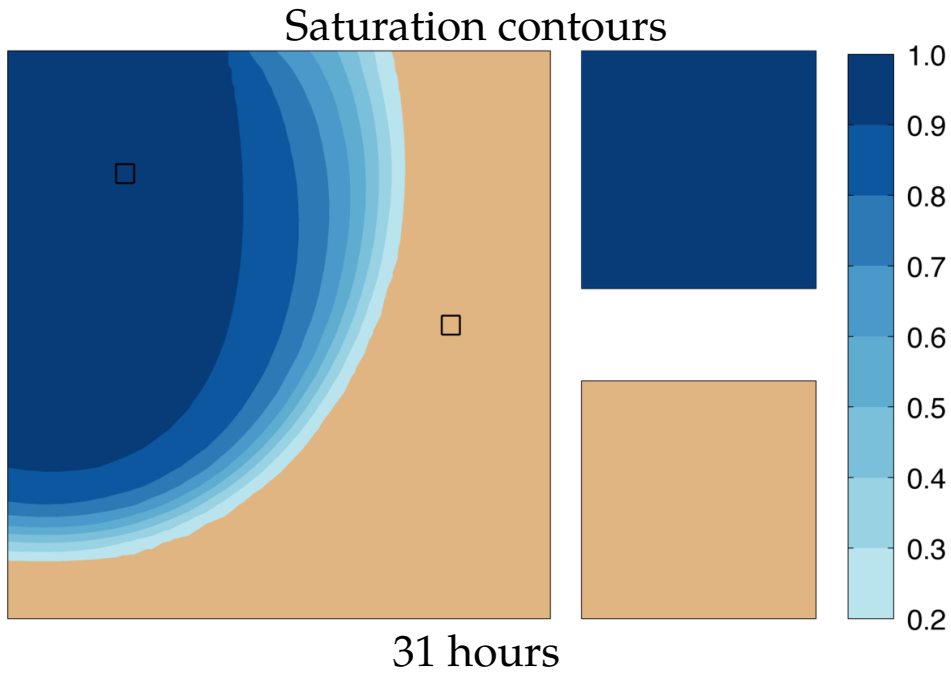
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



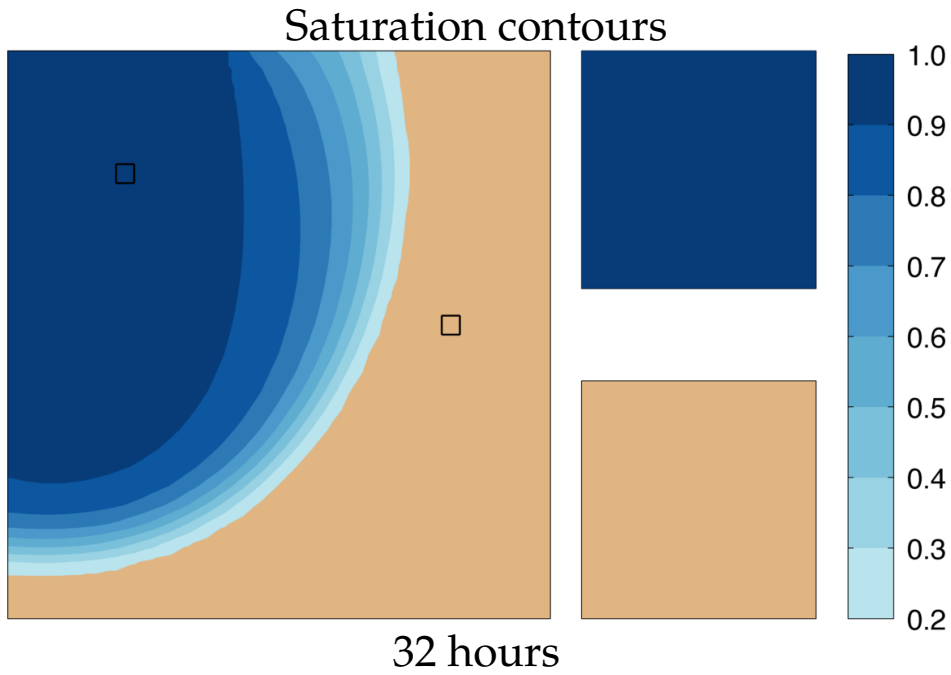
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



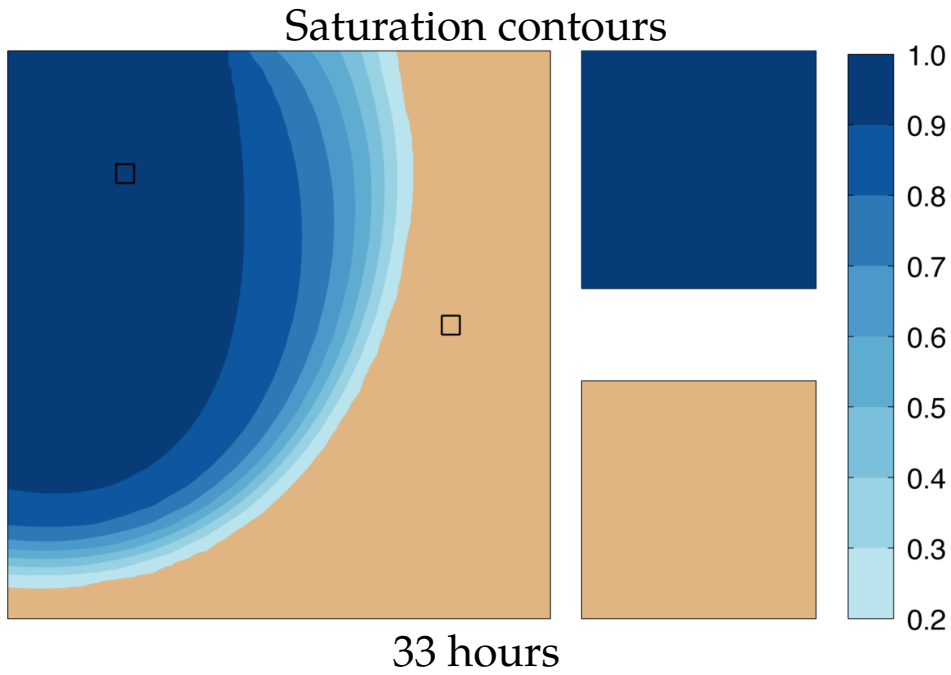
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



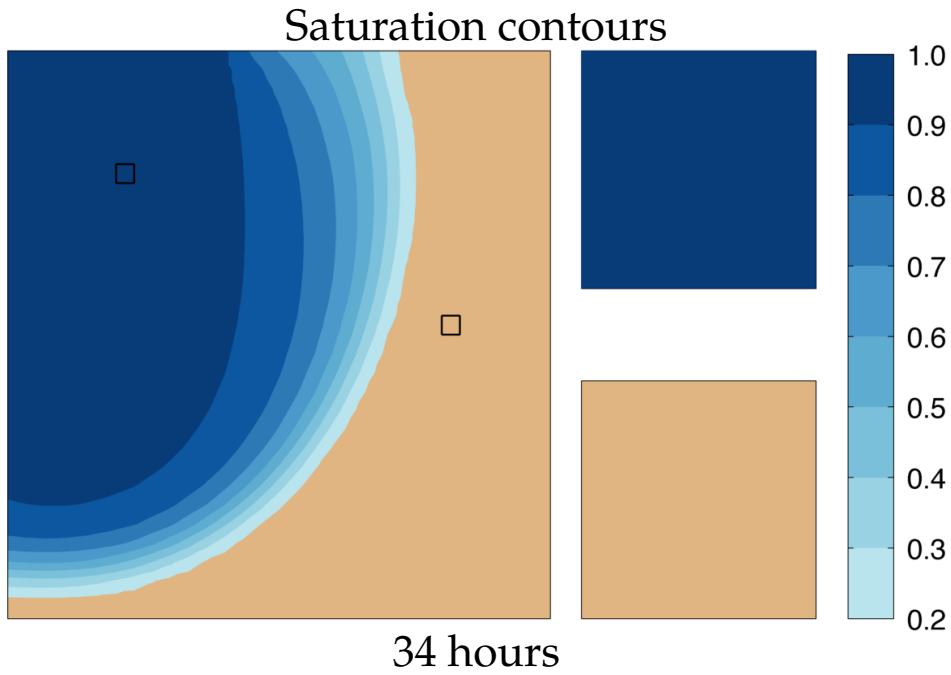
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



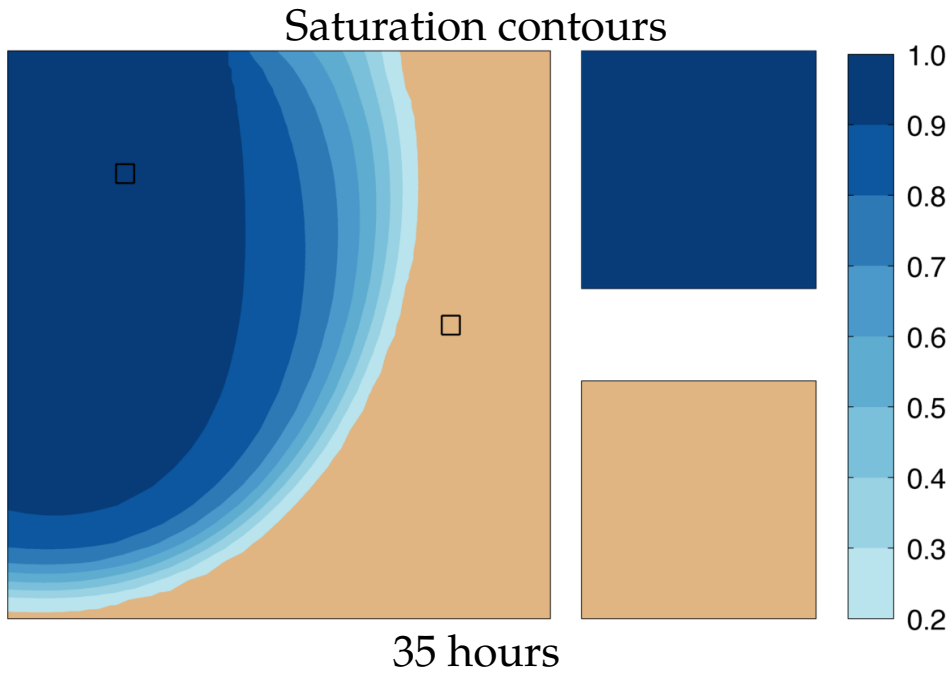
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



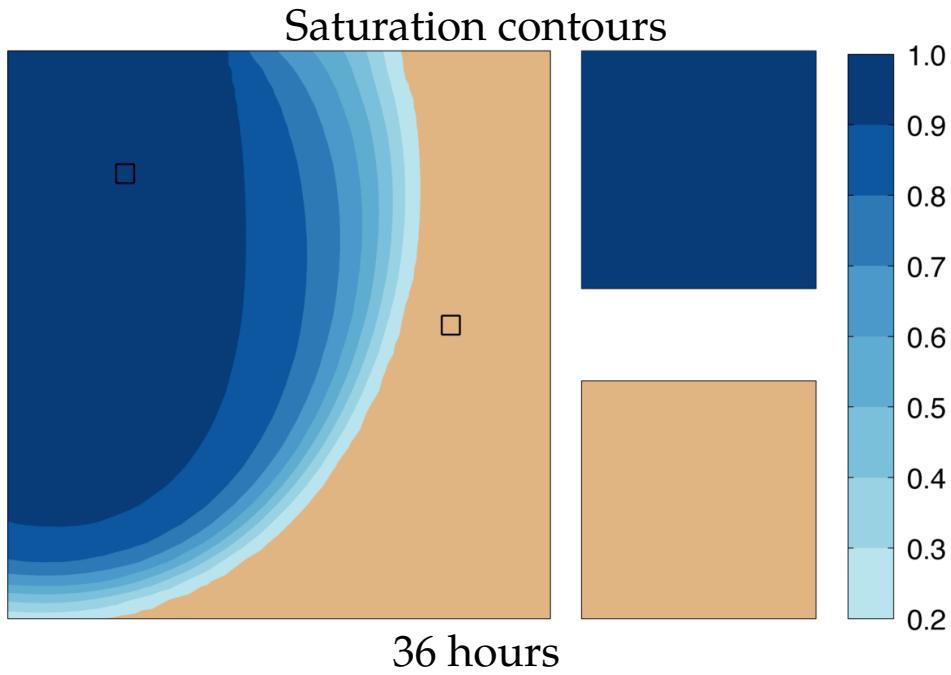
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



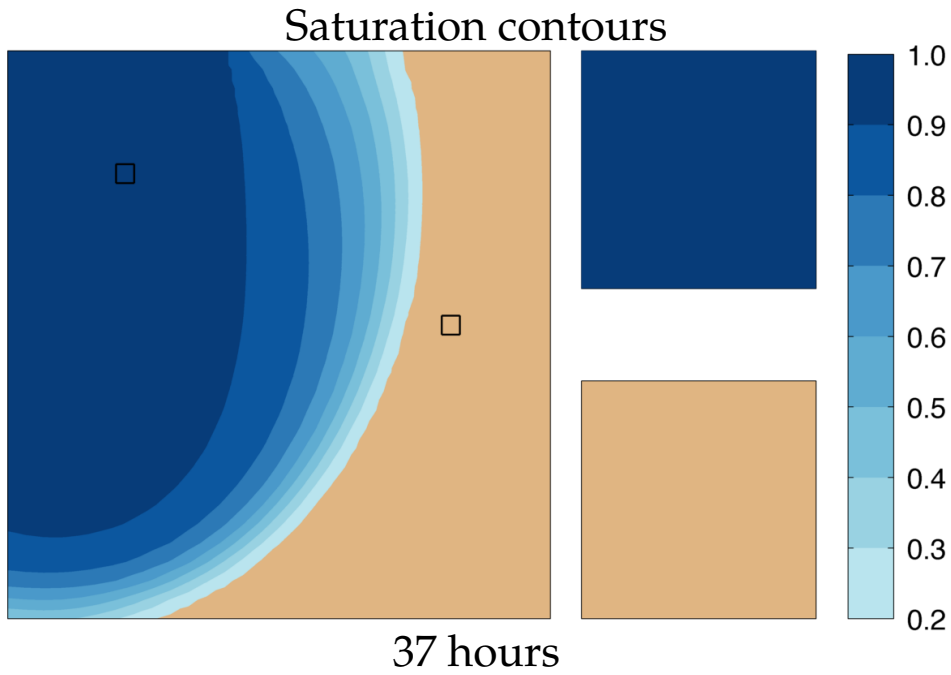
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



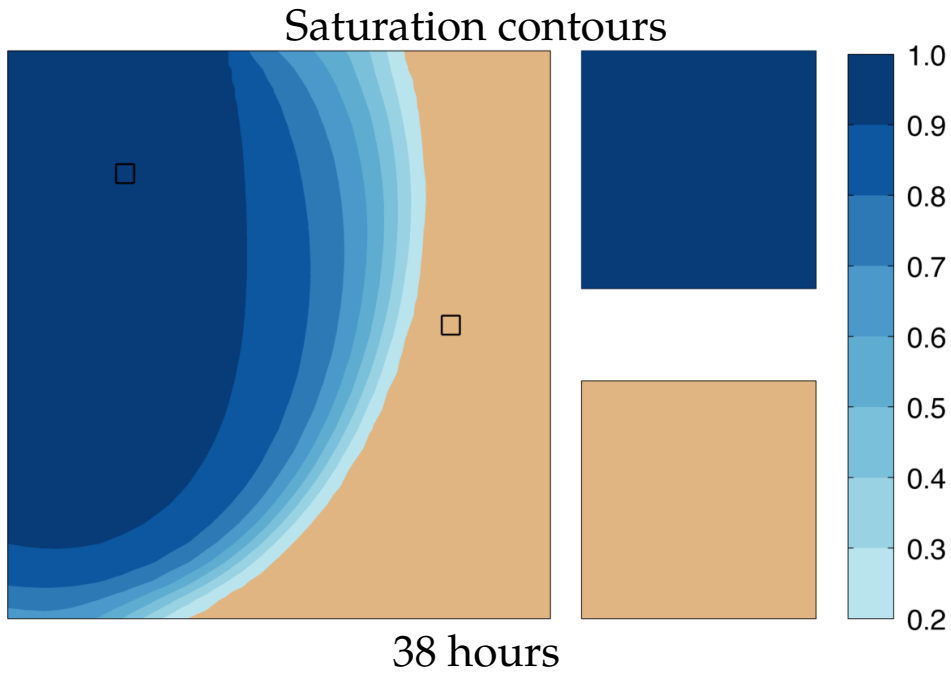
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



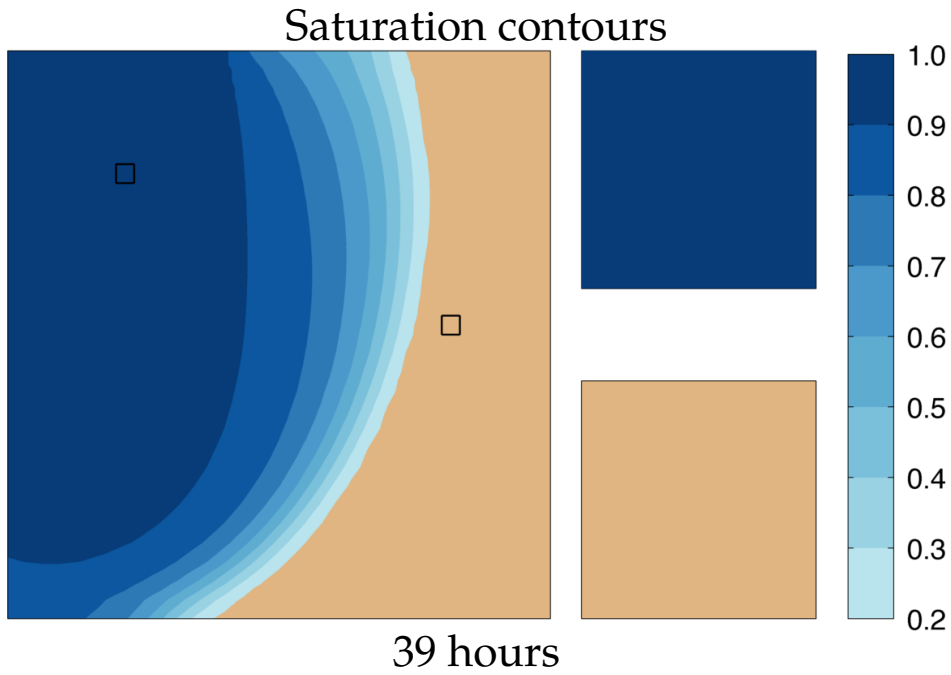
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



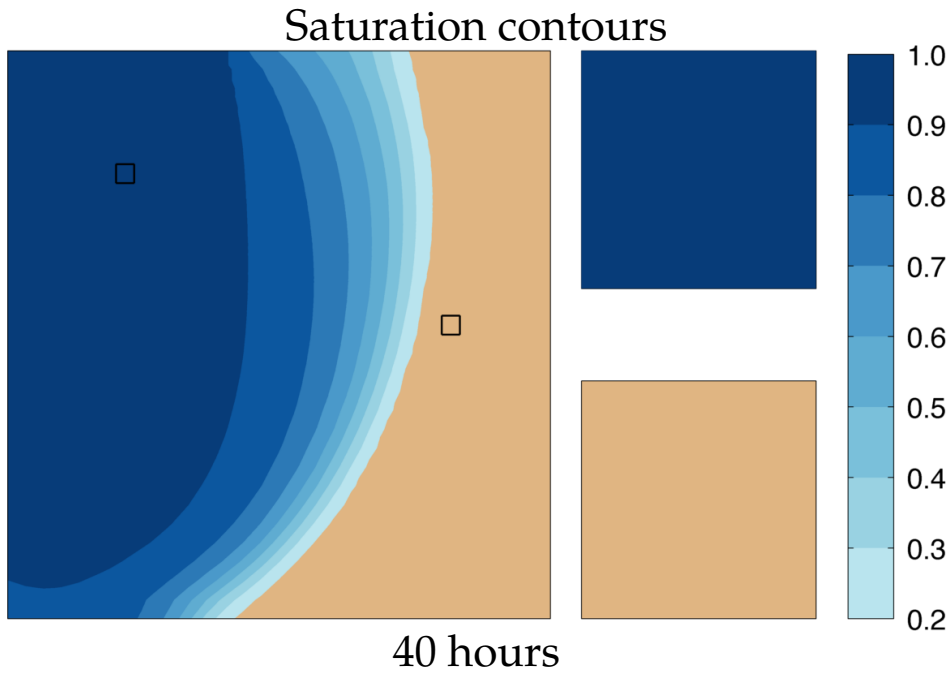
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



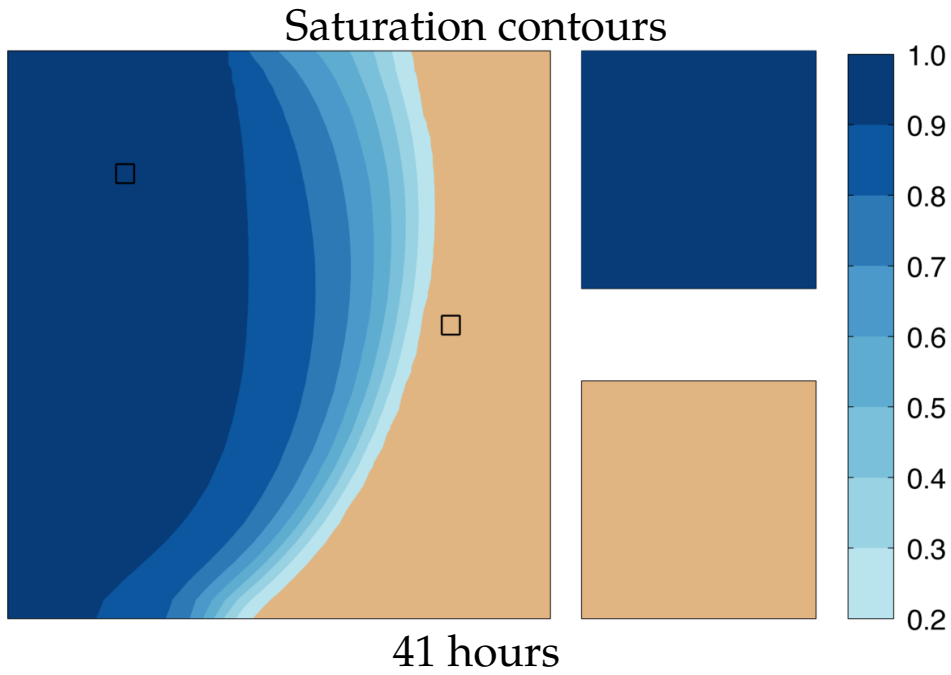
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



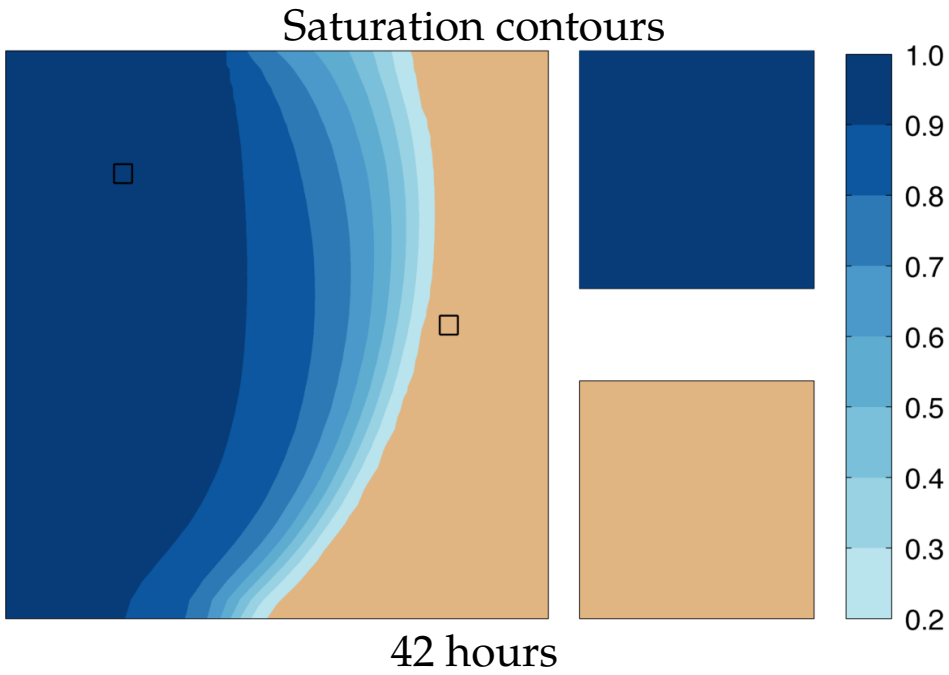
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



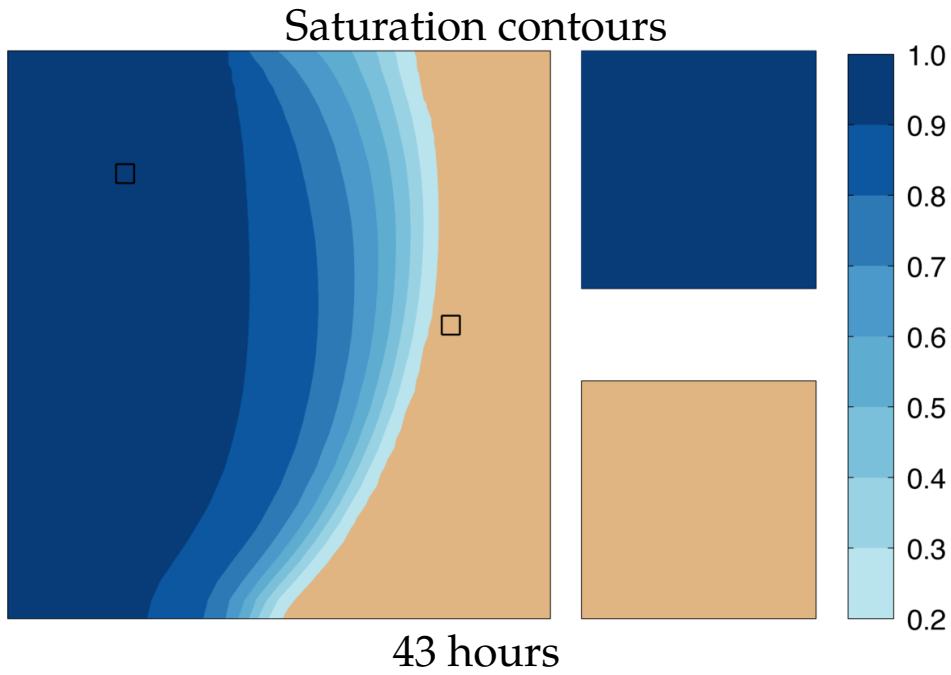
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



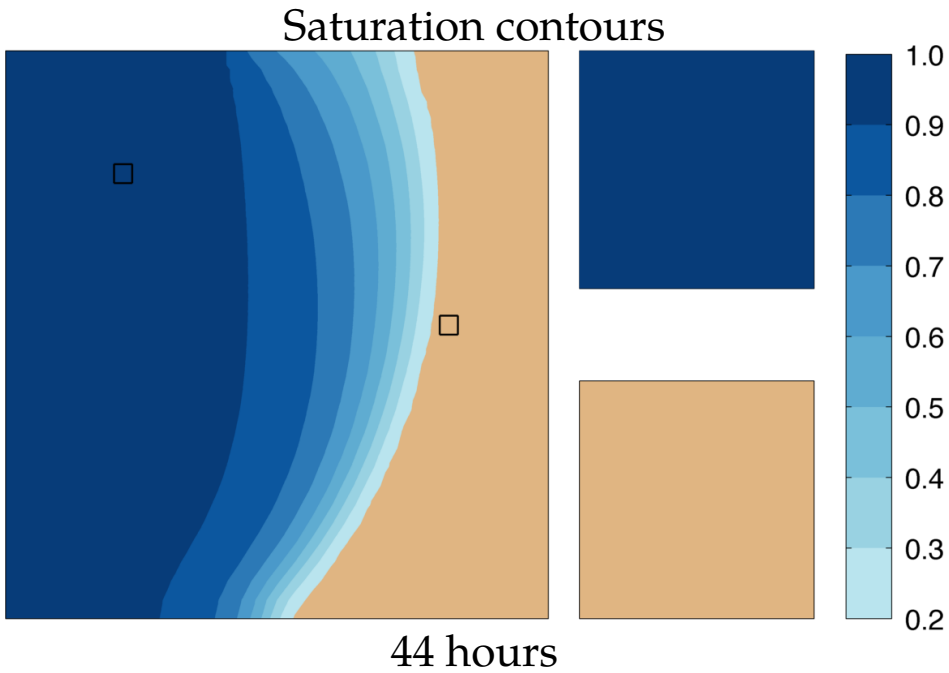
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



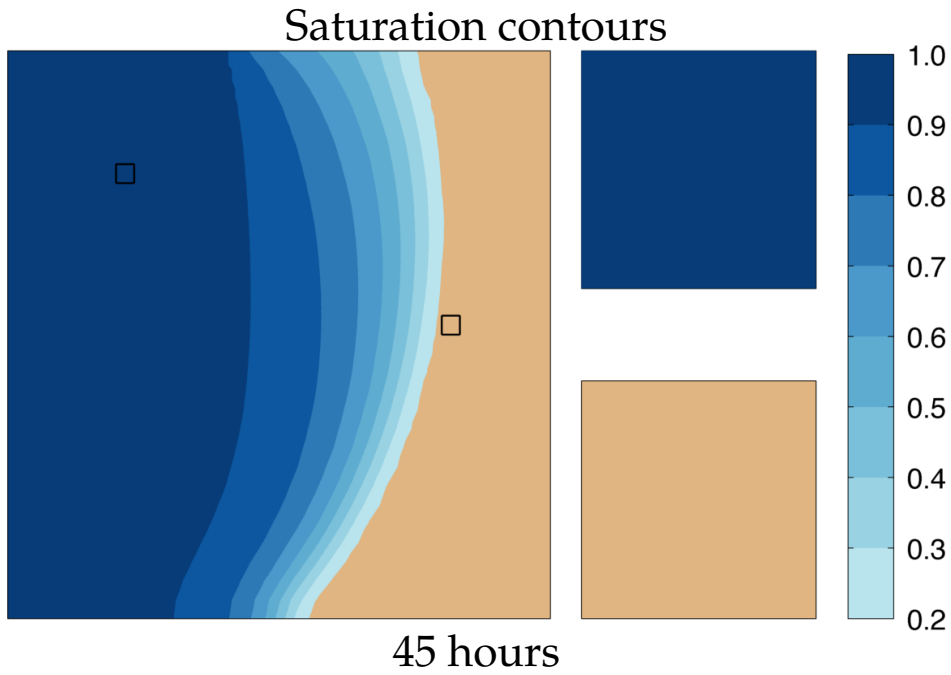
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



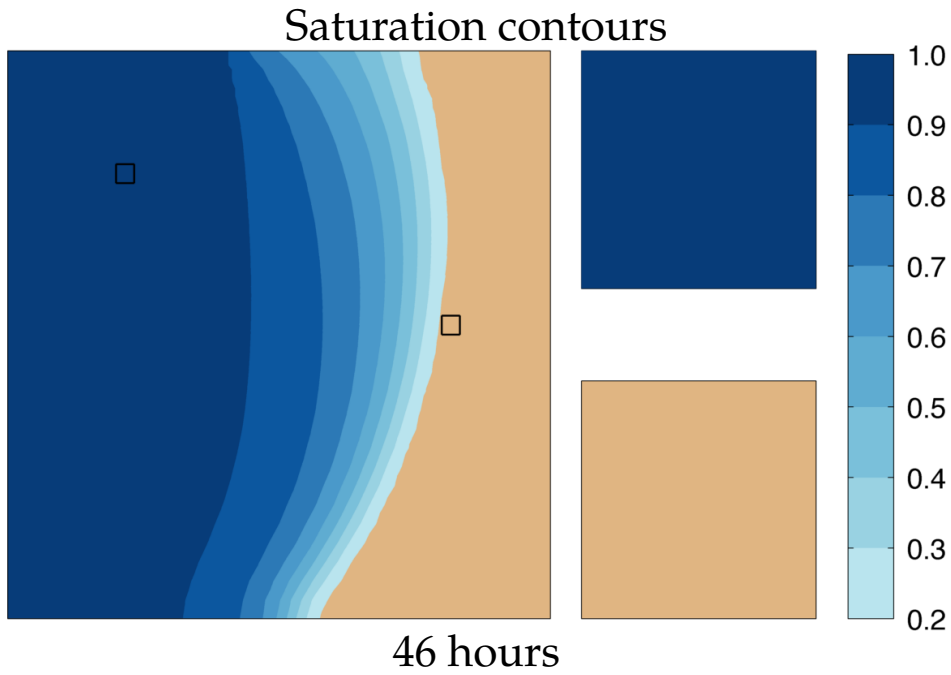
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



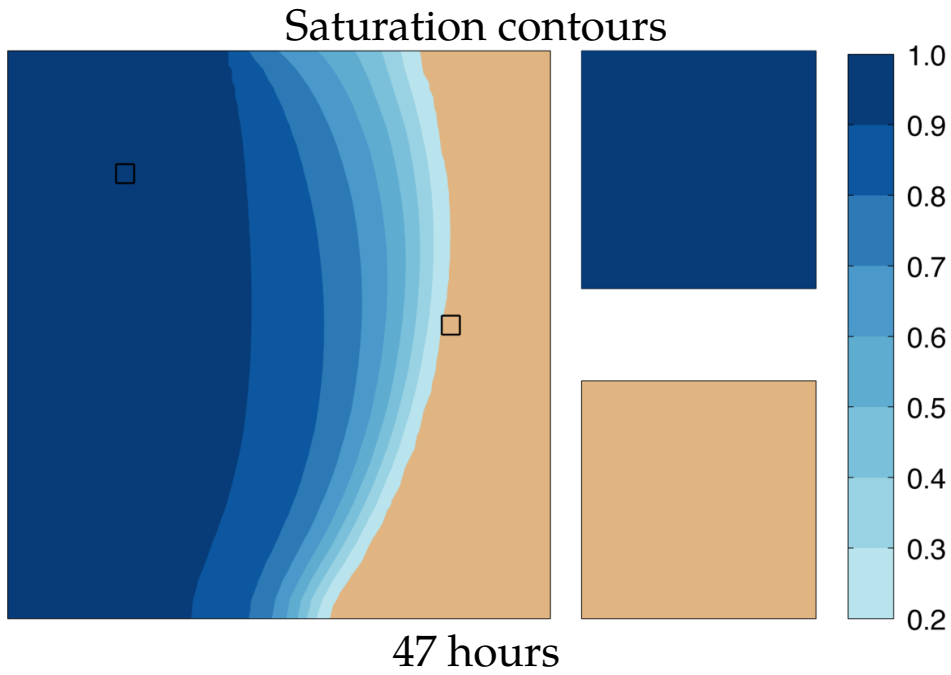
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



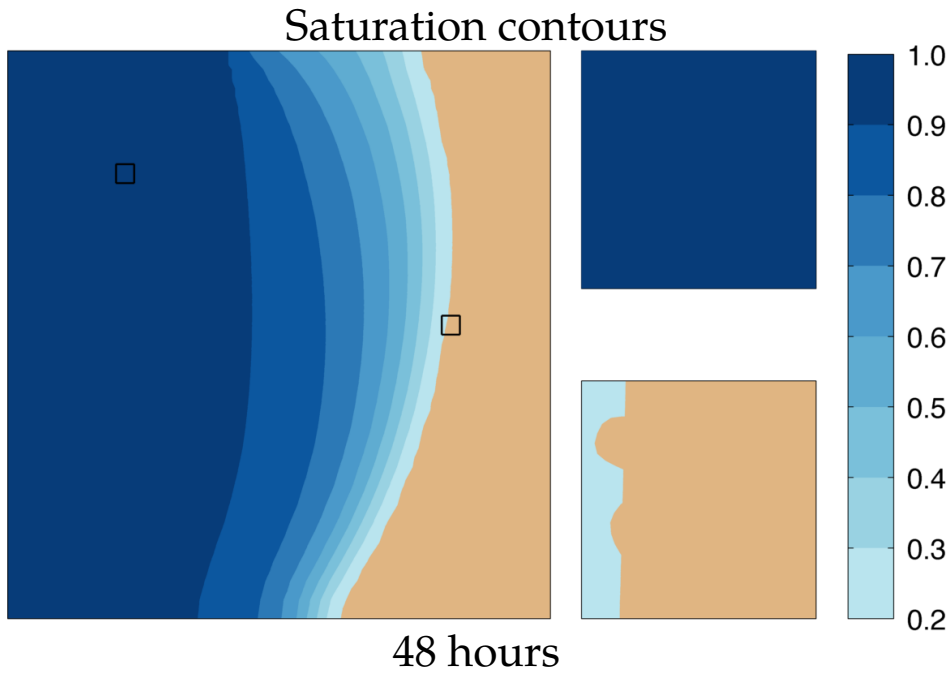
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



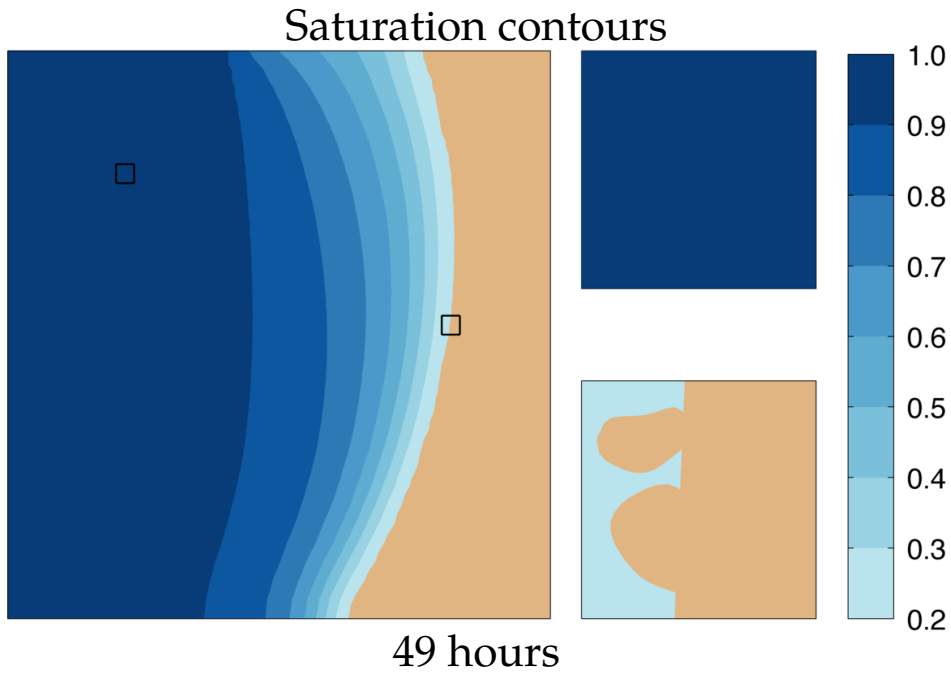
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



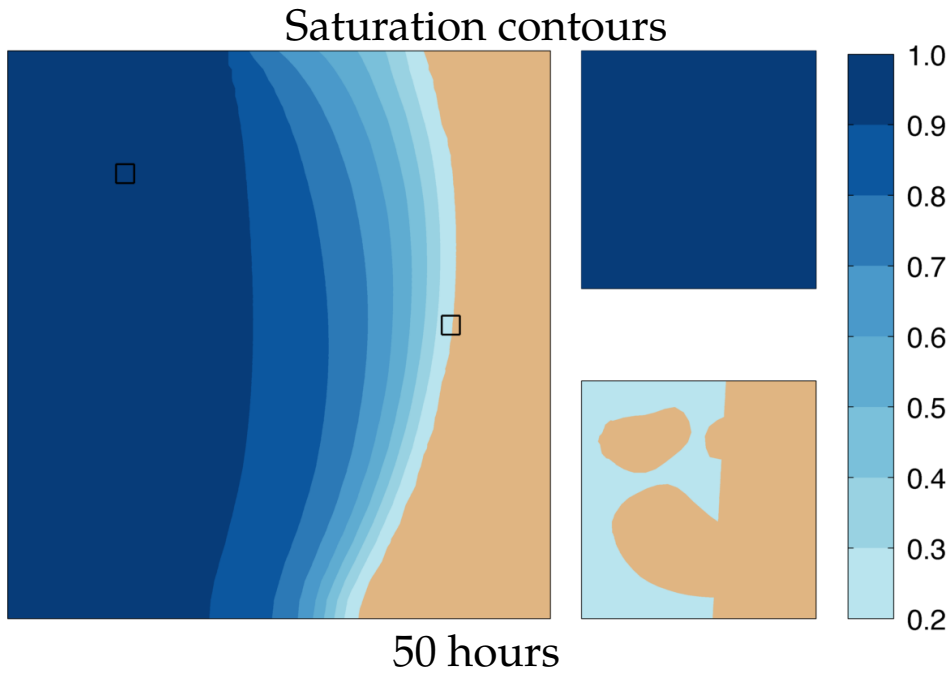
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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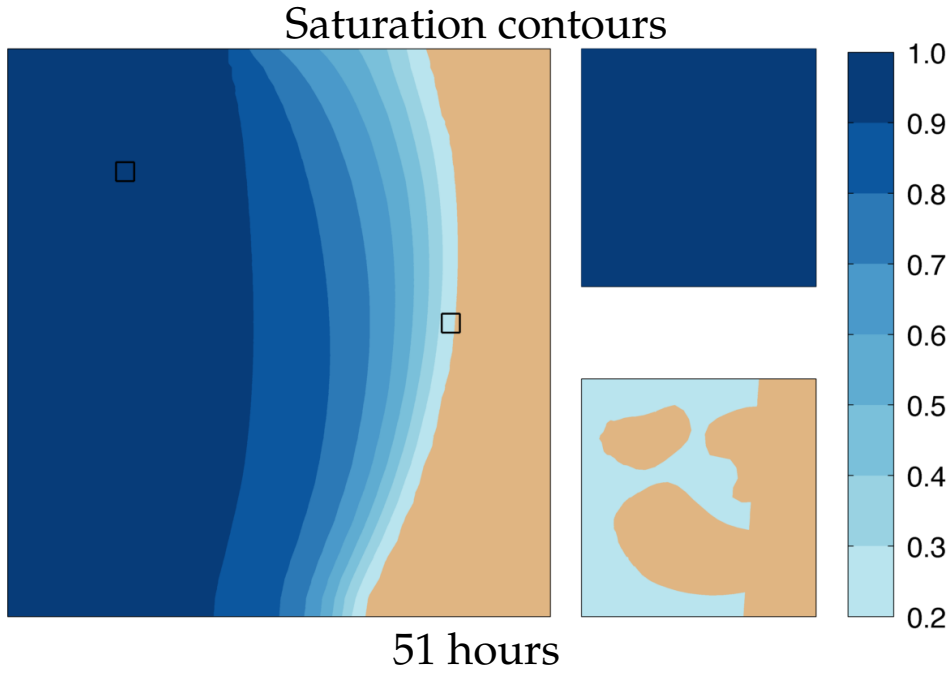
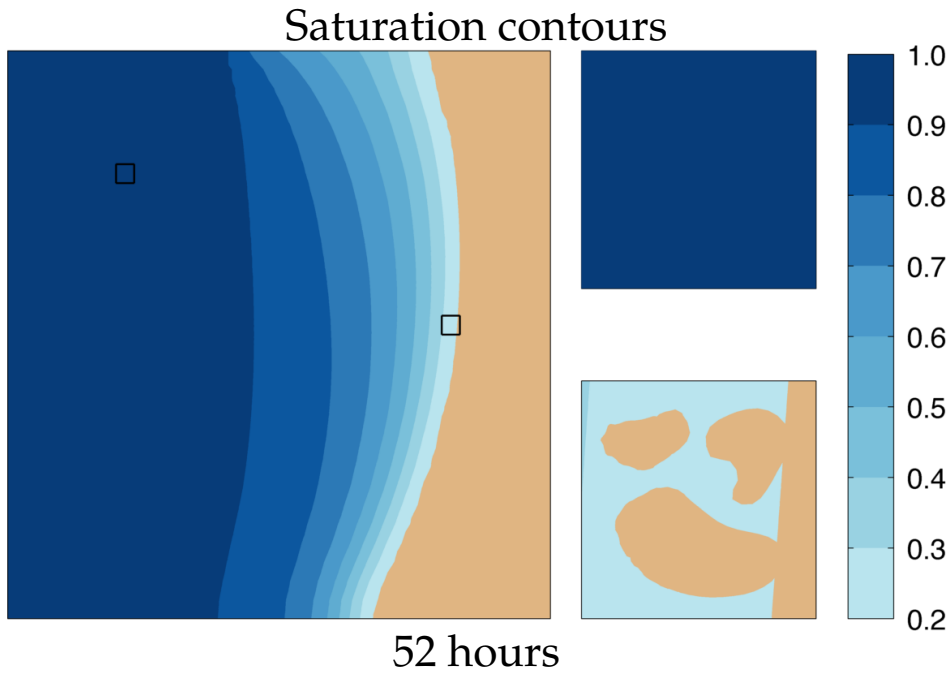
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



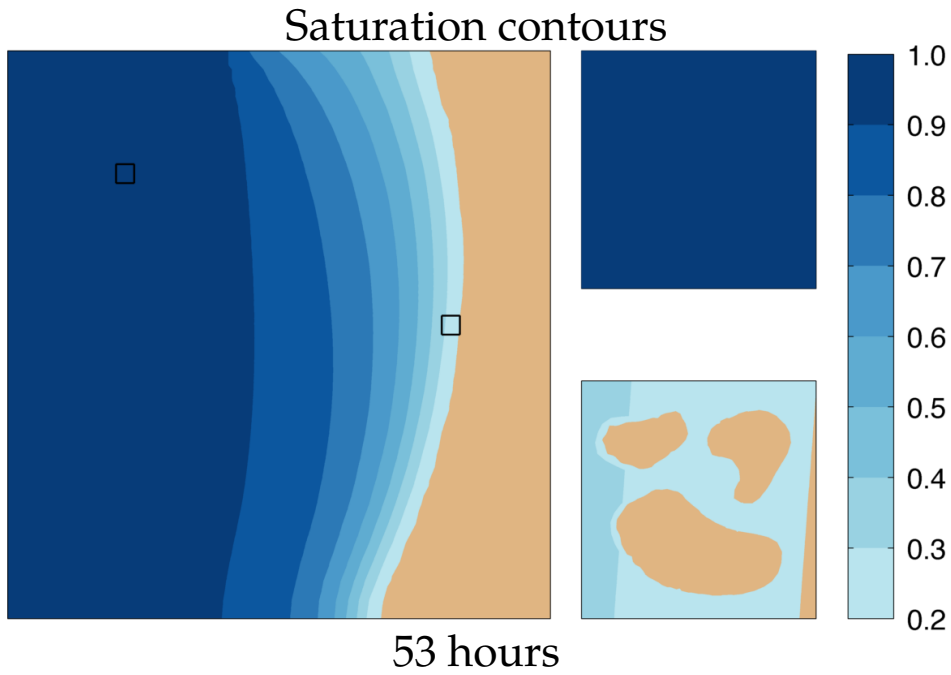
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



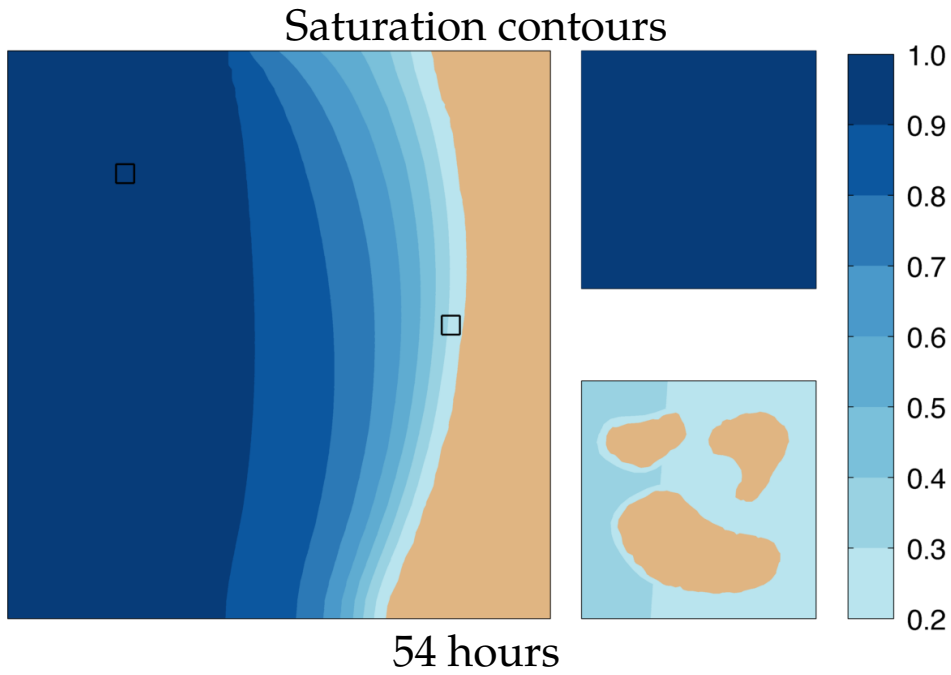
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



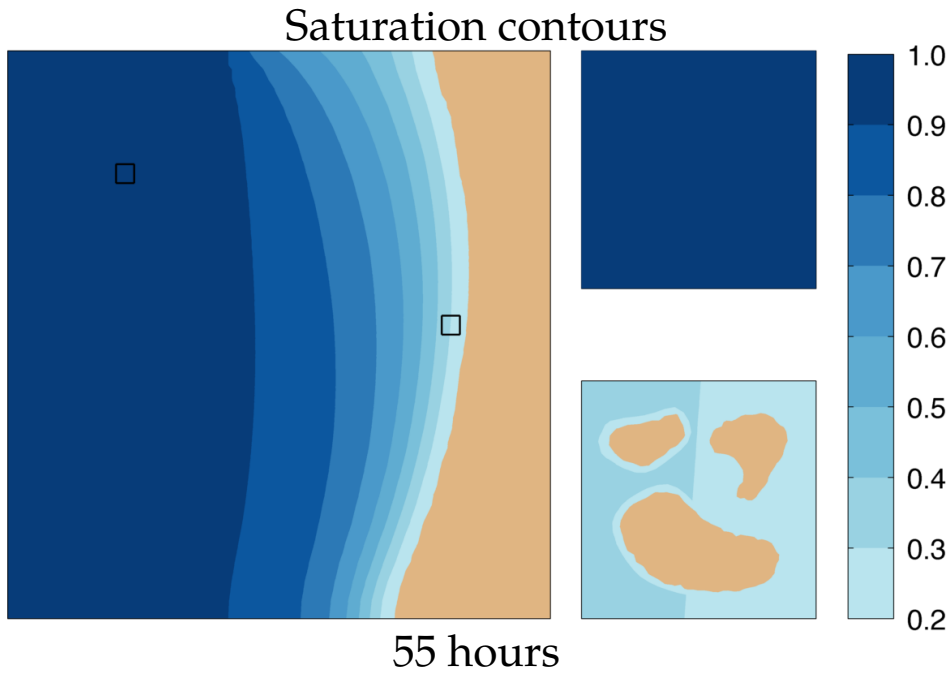
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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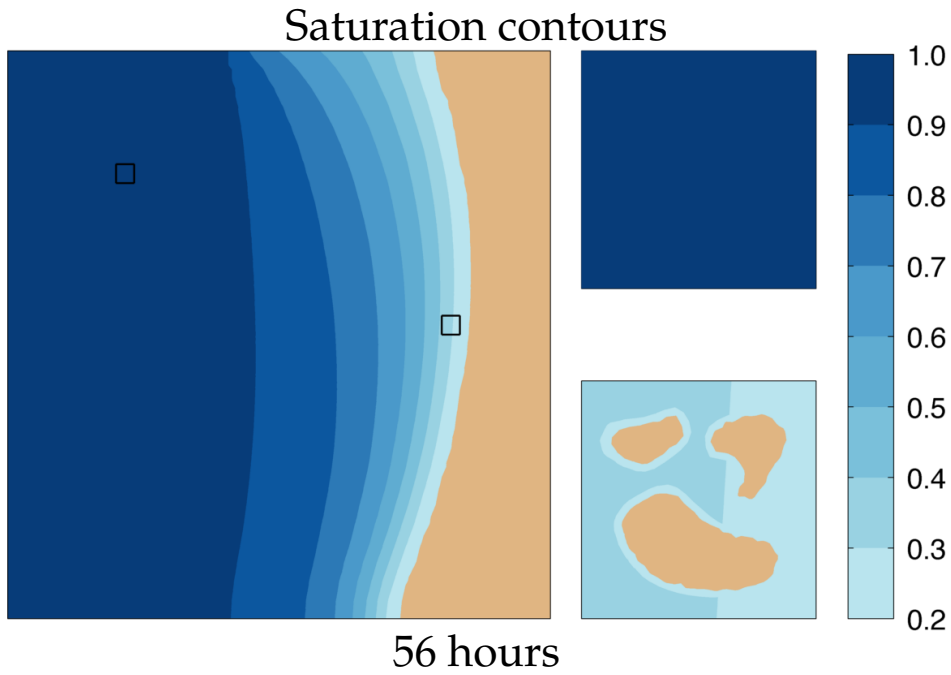
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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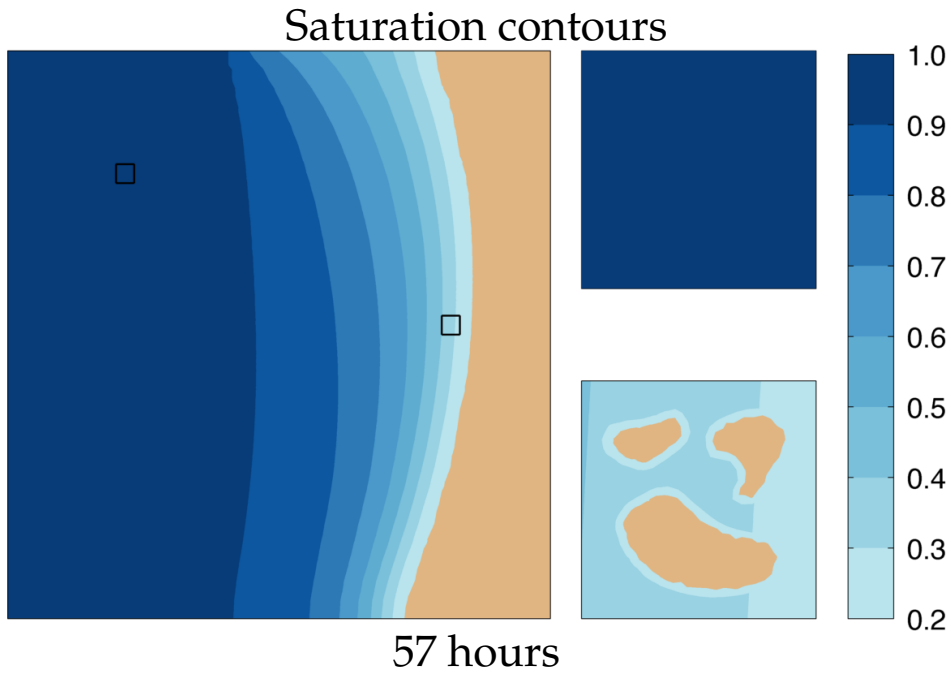
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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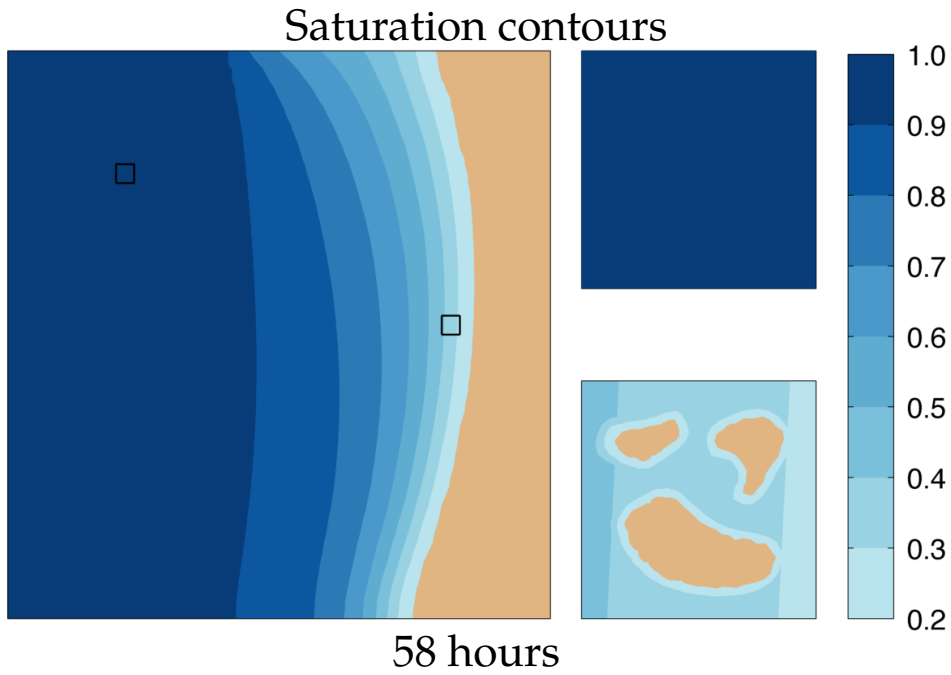
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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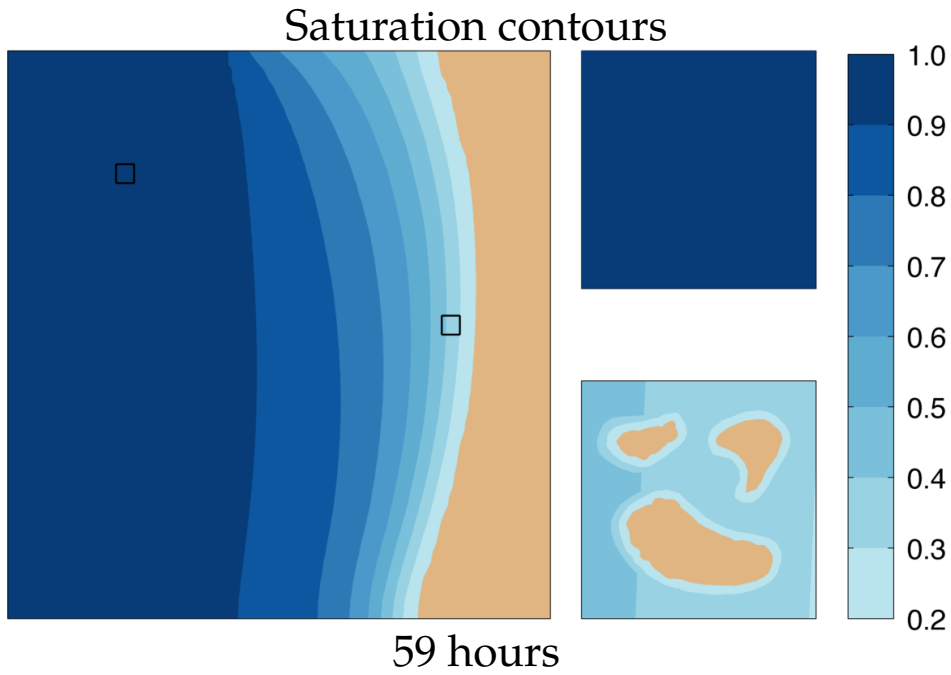
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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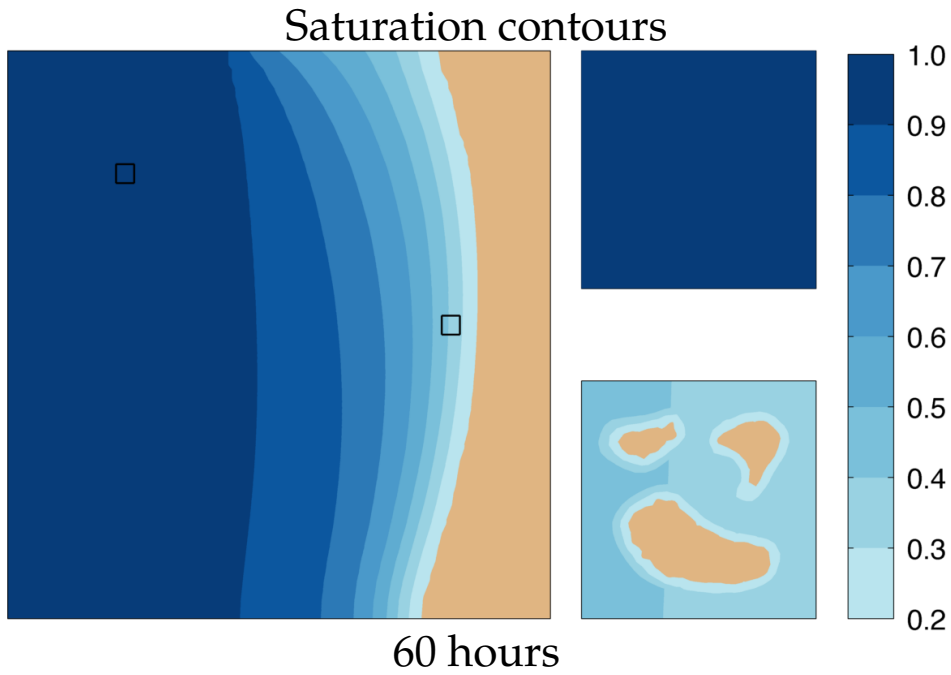
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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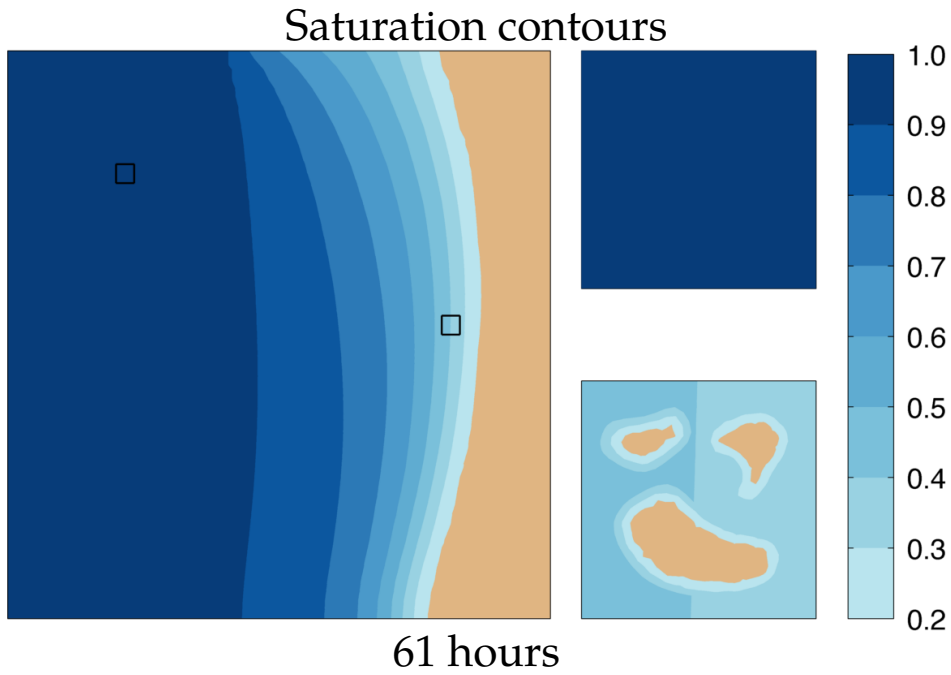
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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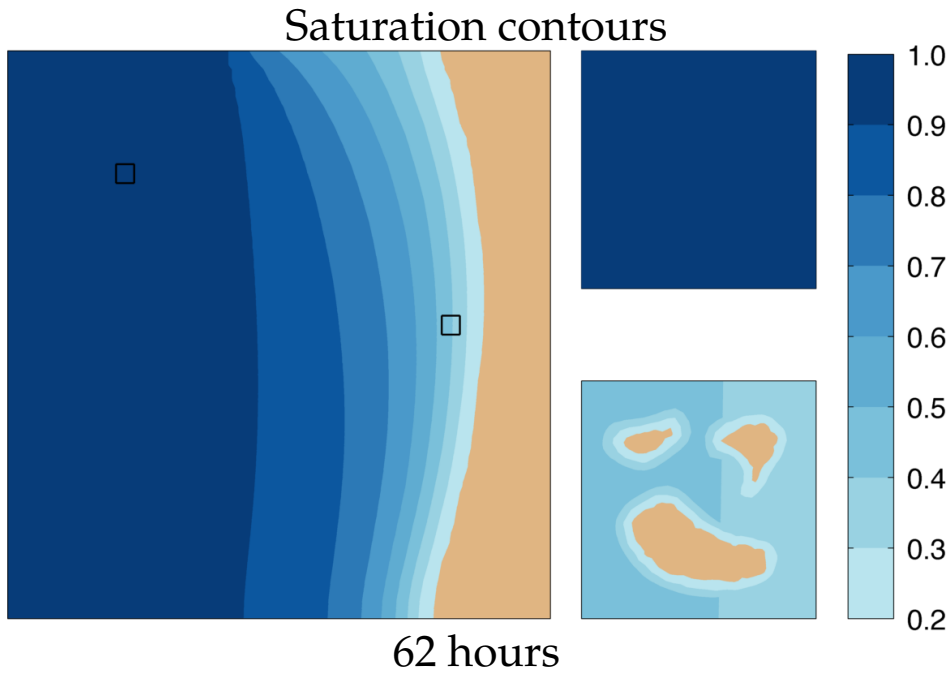
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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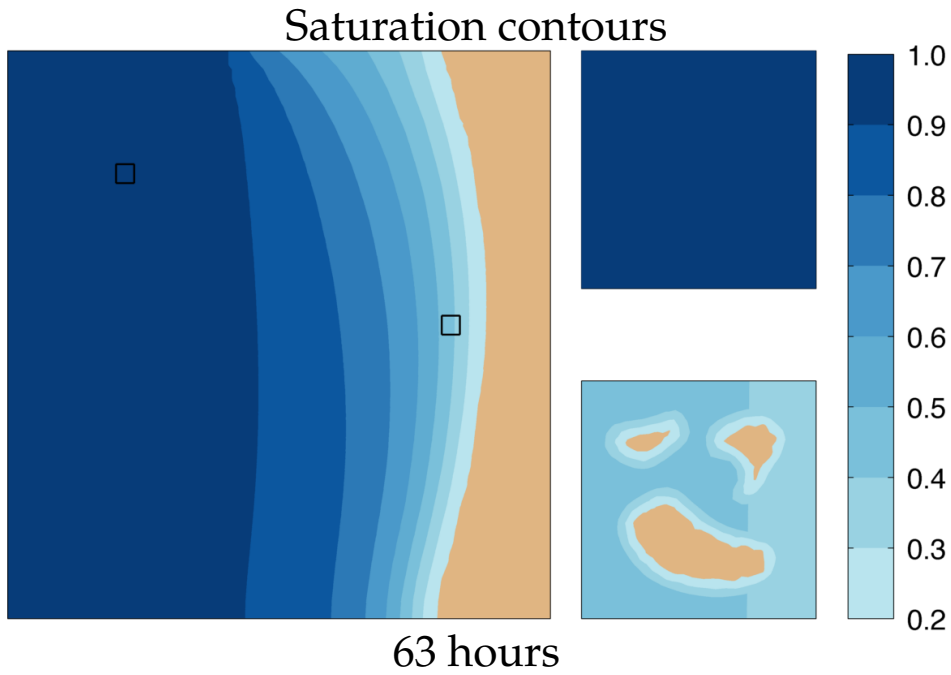
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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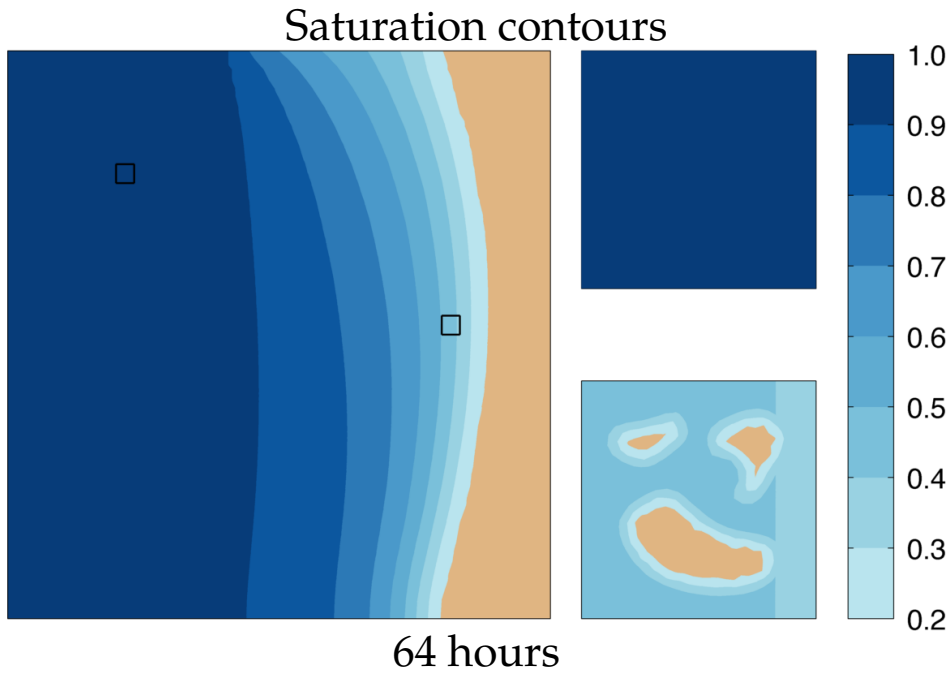
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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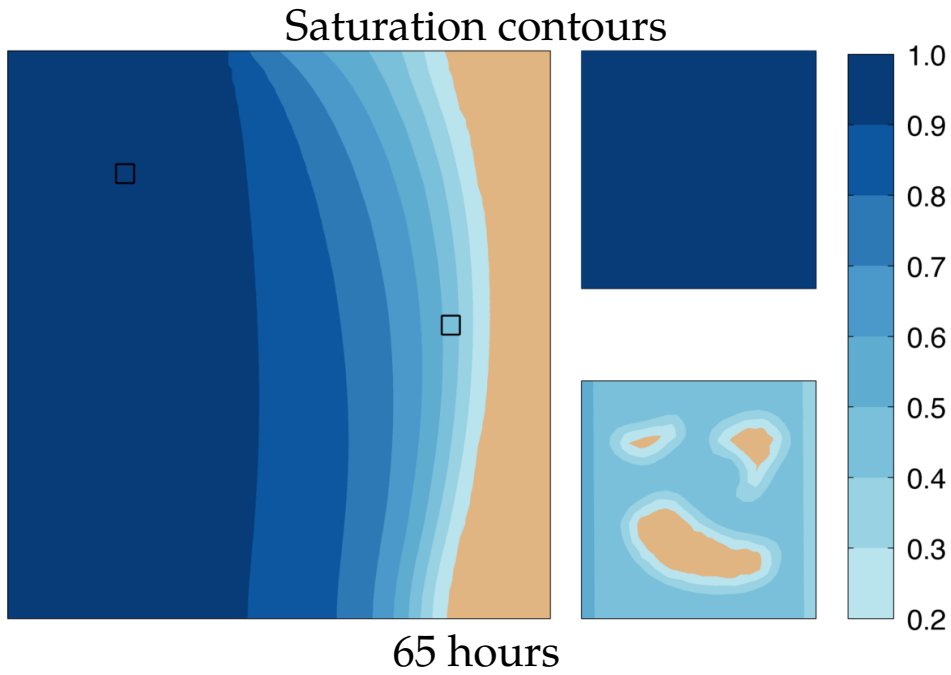
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
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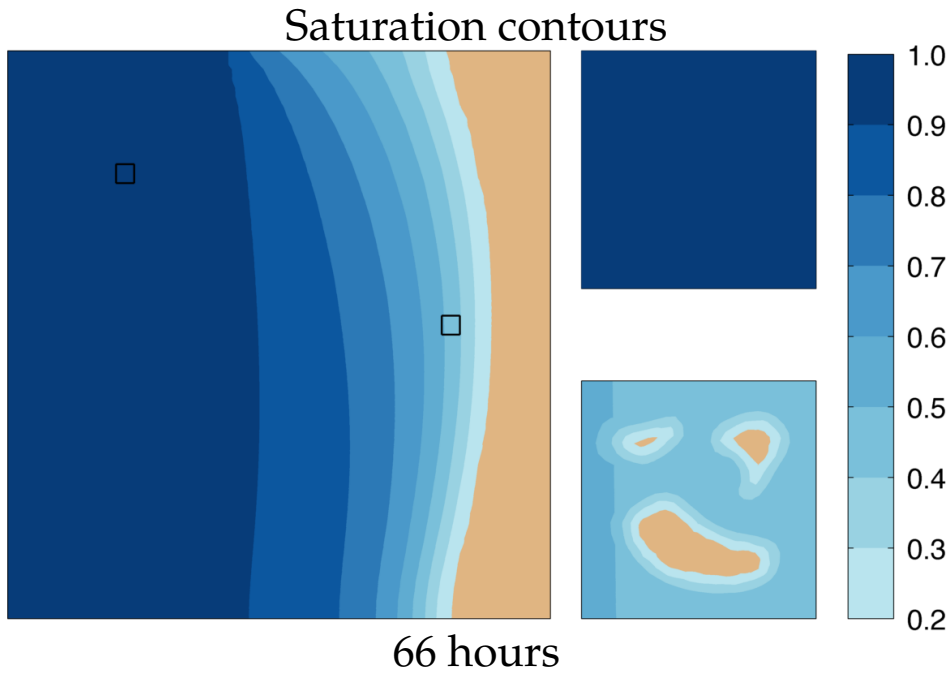
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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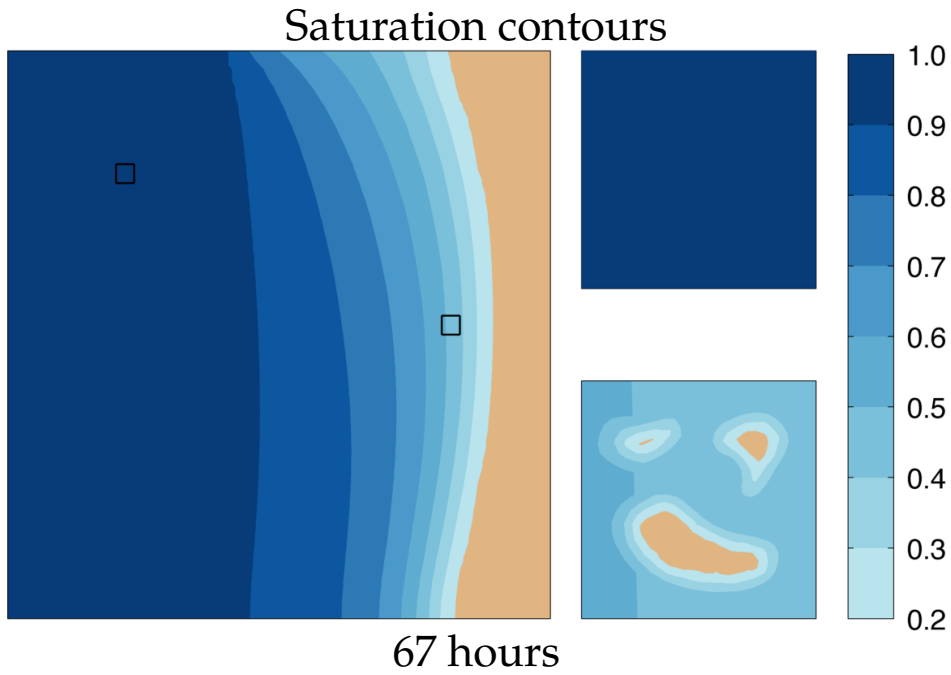
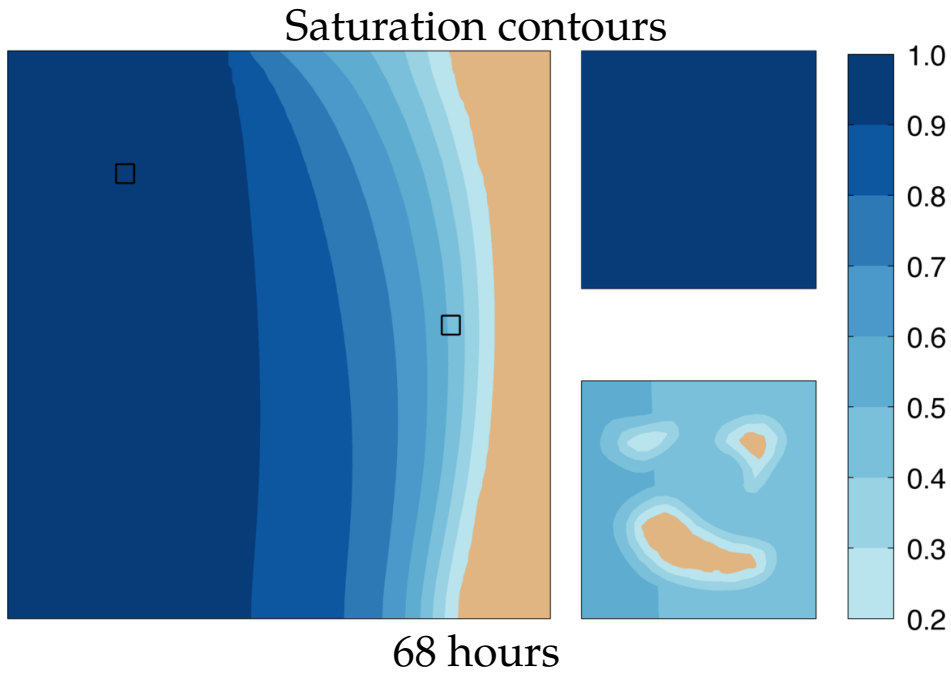
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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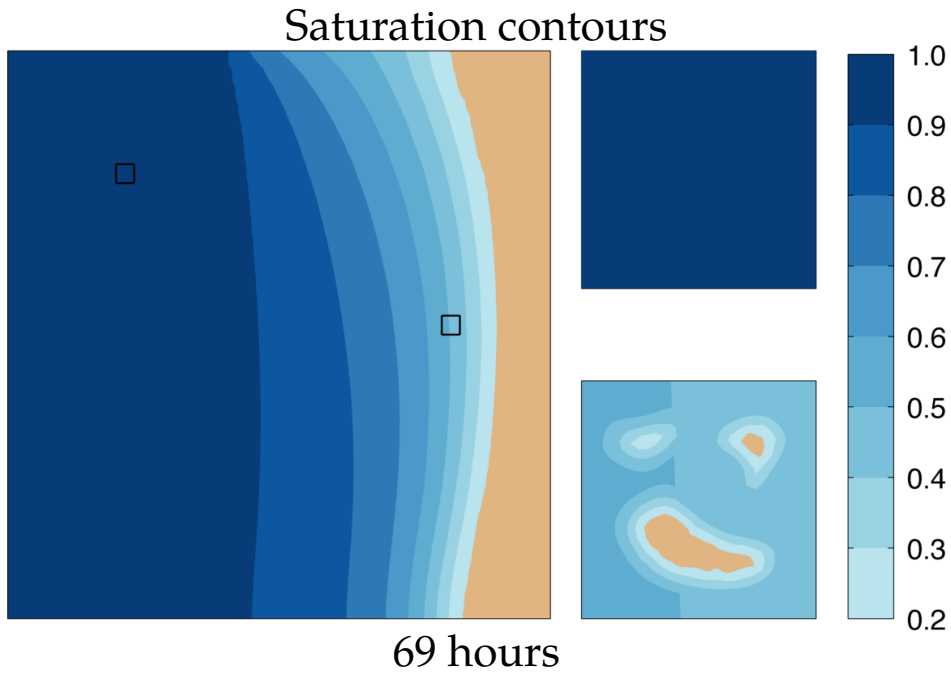
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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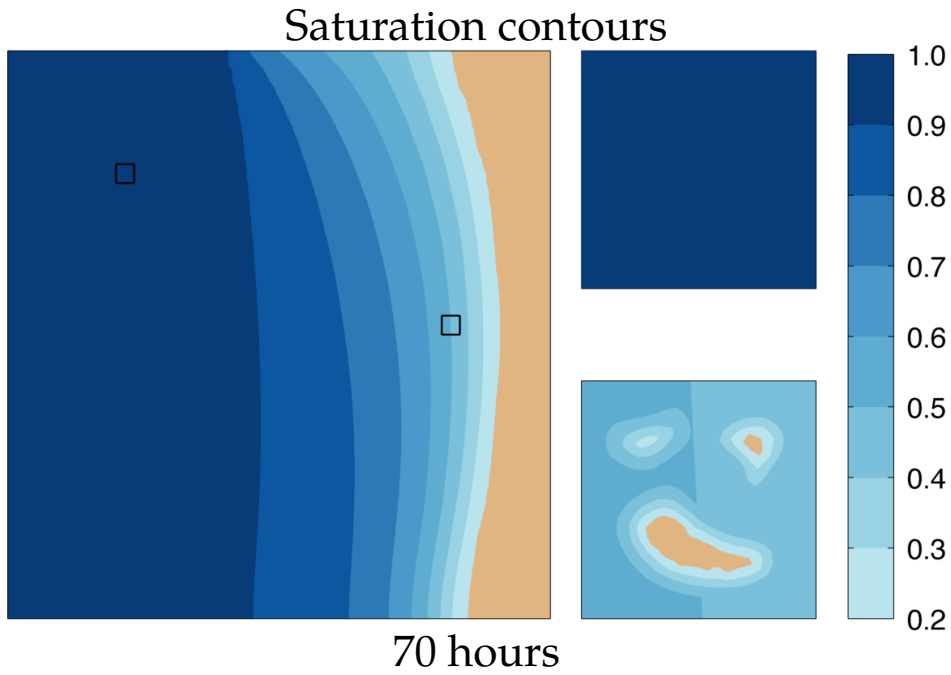
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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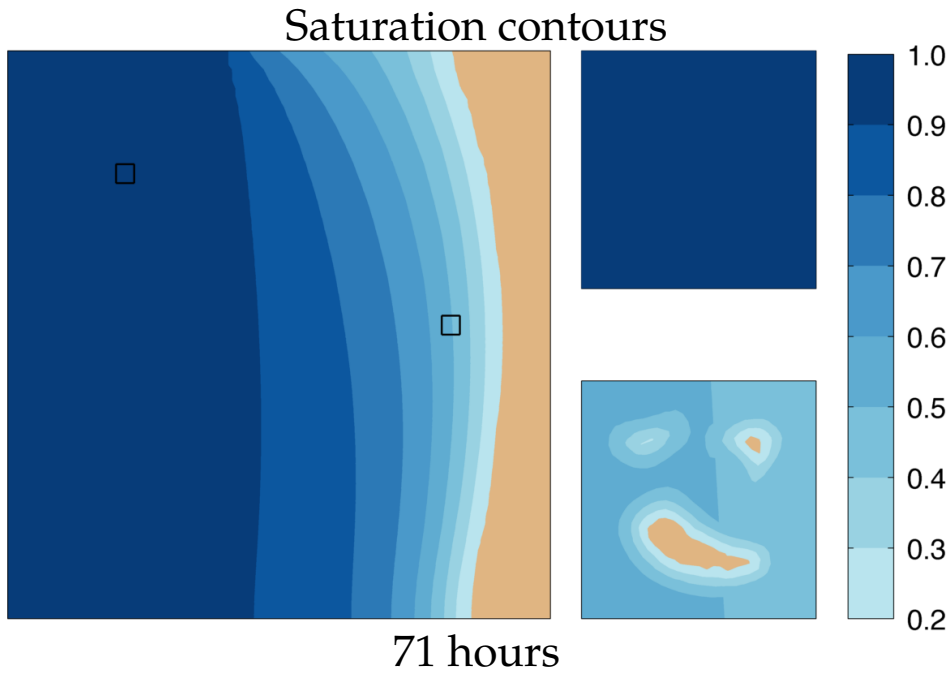
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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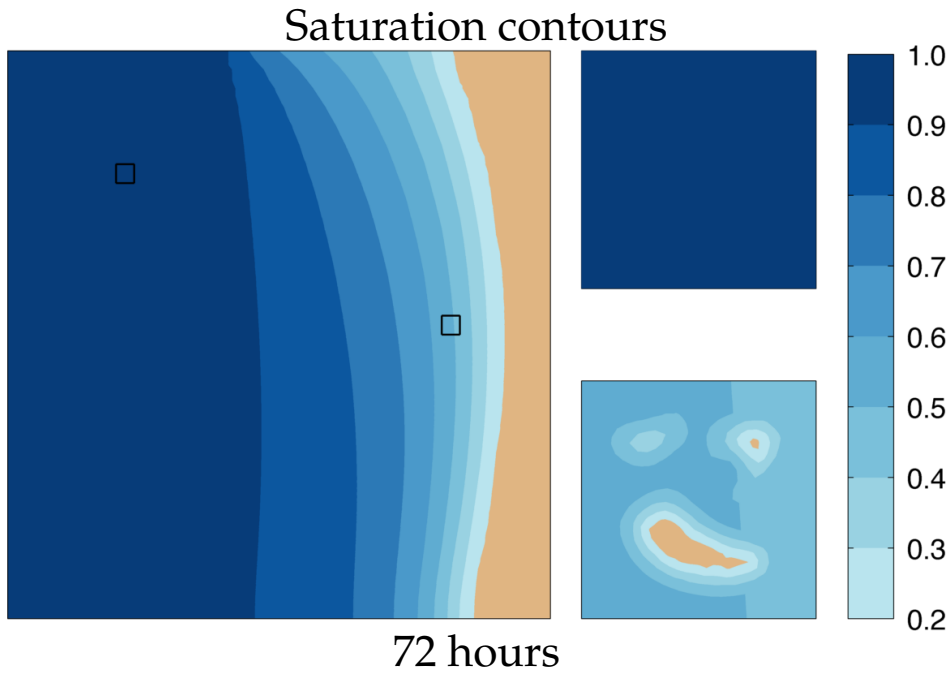
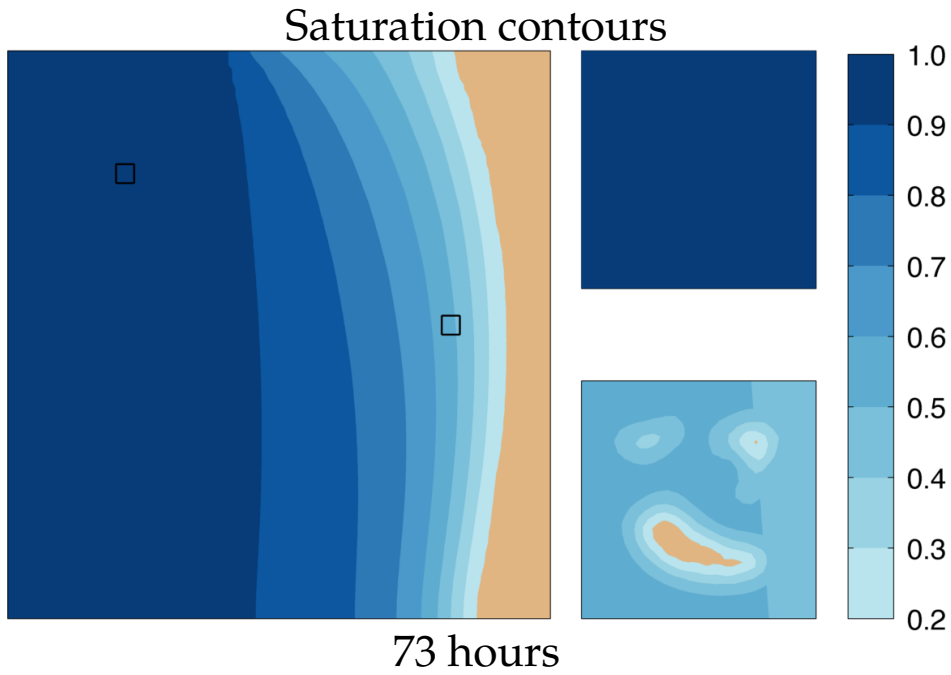
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



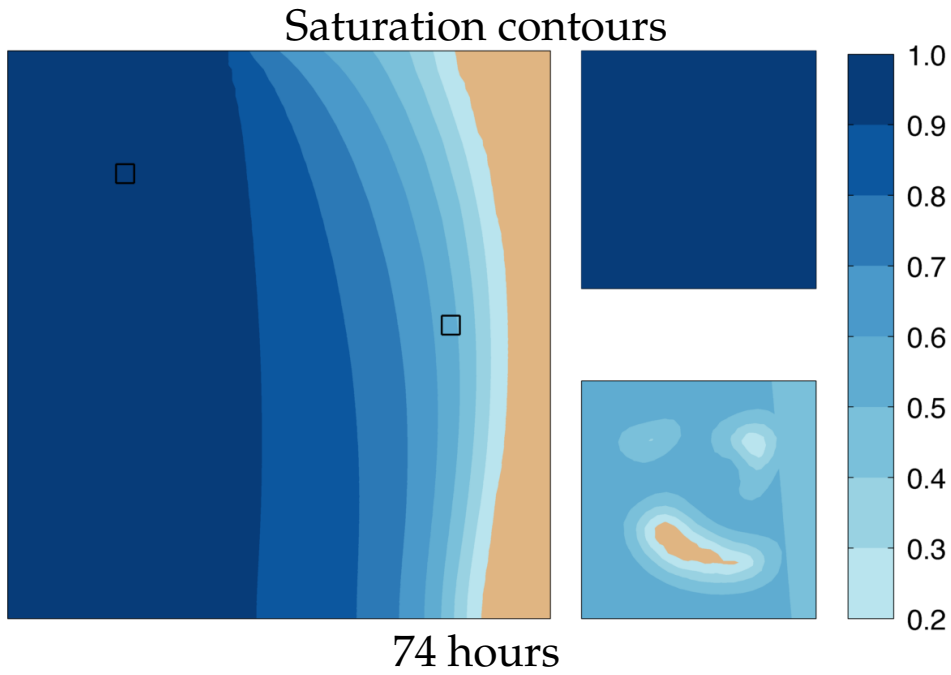
	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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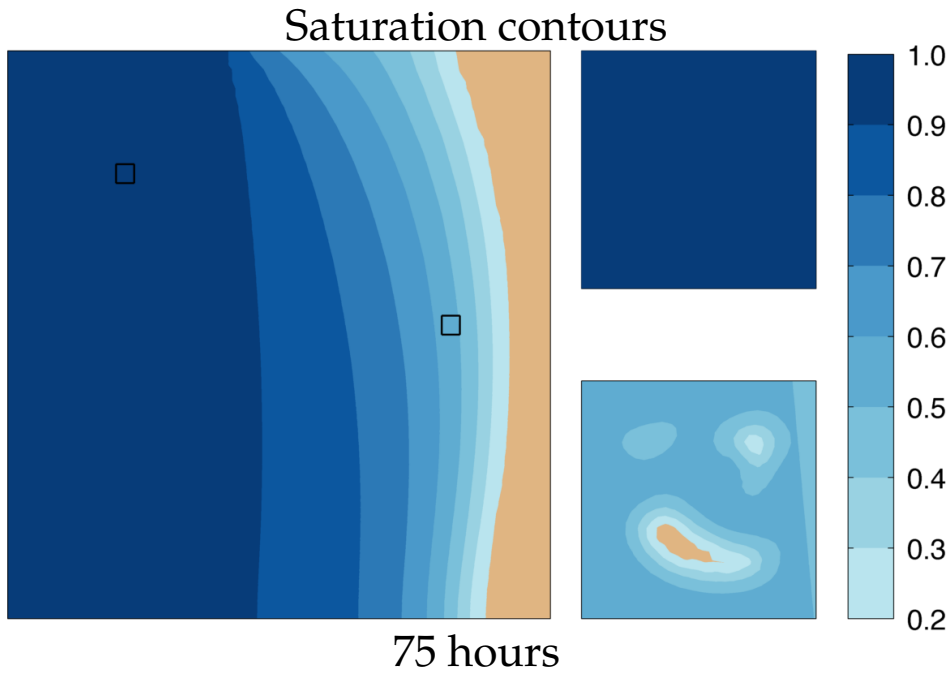
Simulation: Unsaturated water flow



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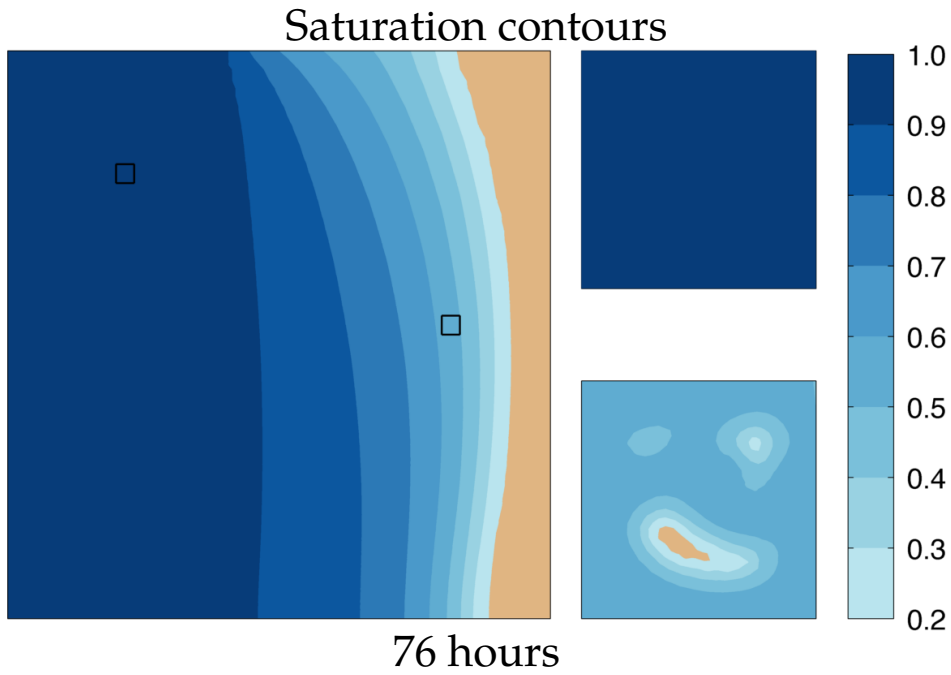
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



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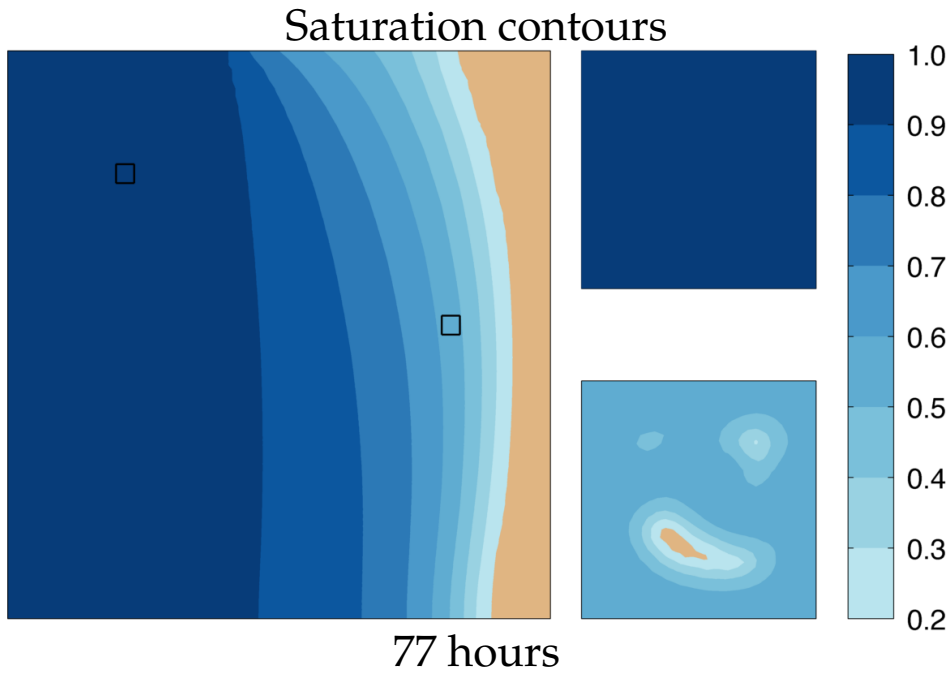
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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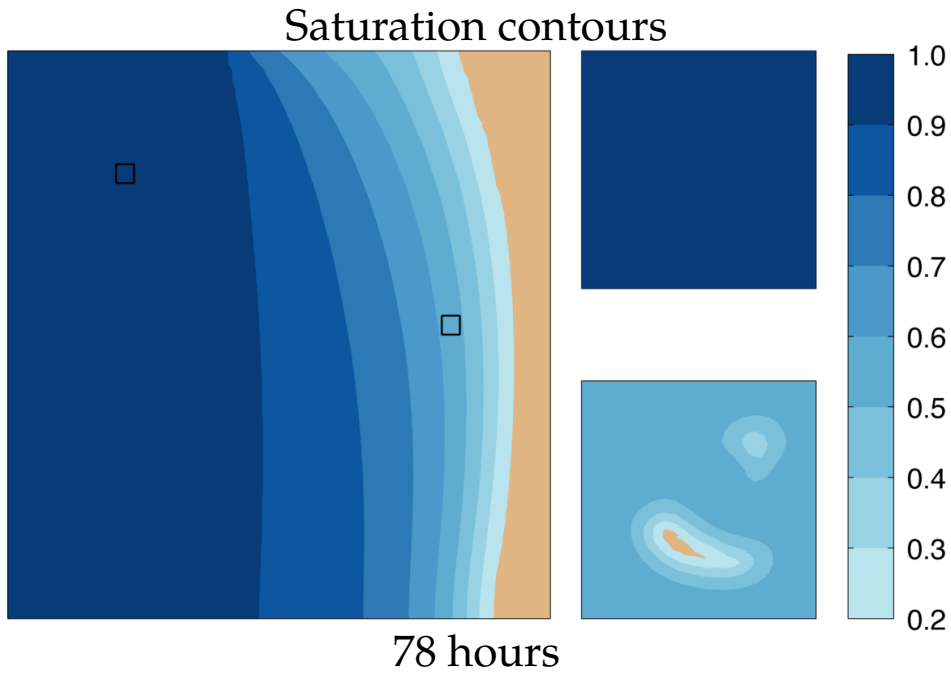
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



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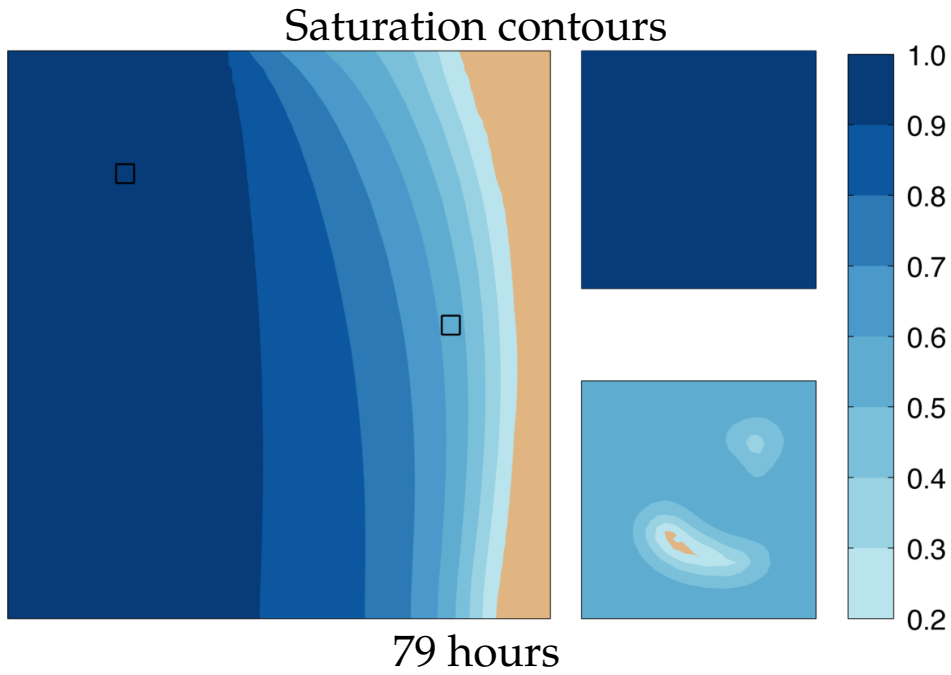
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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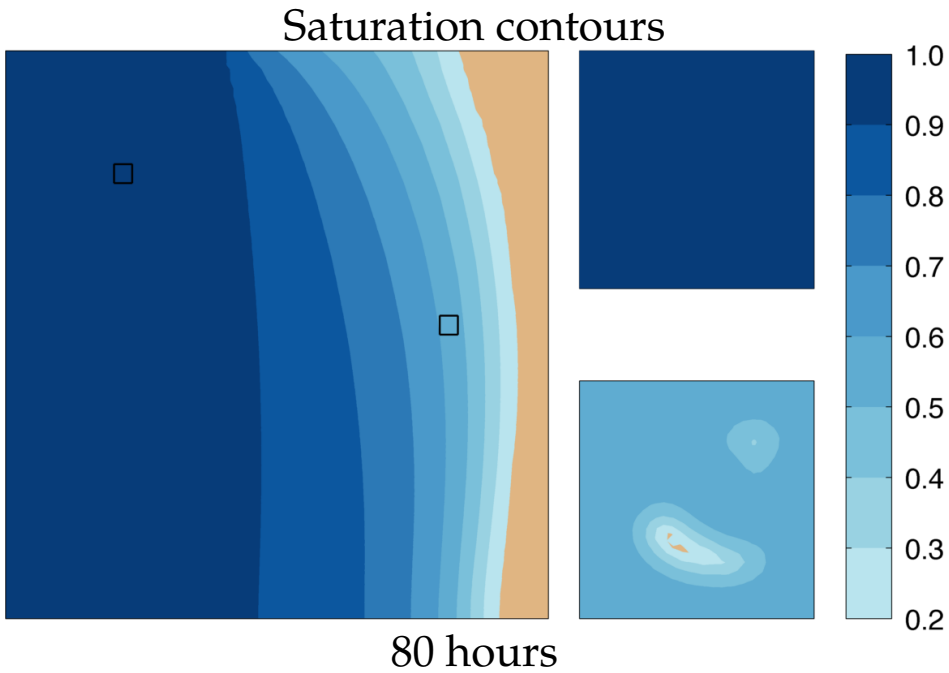
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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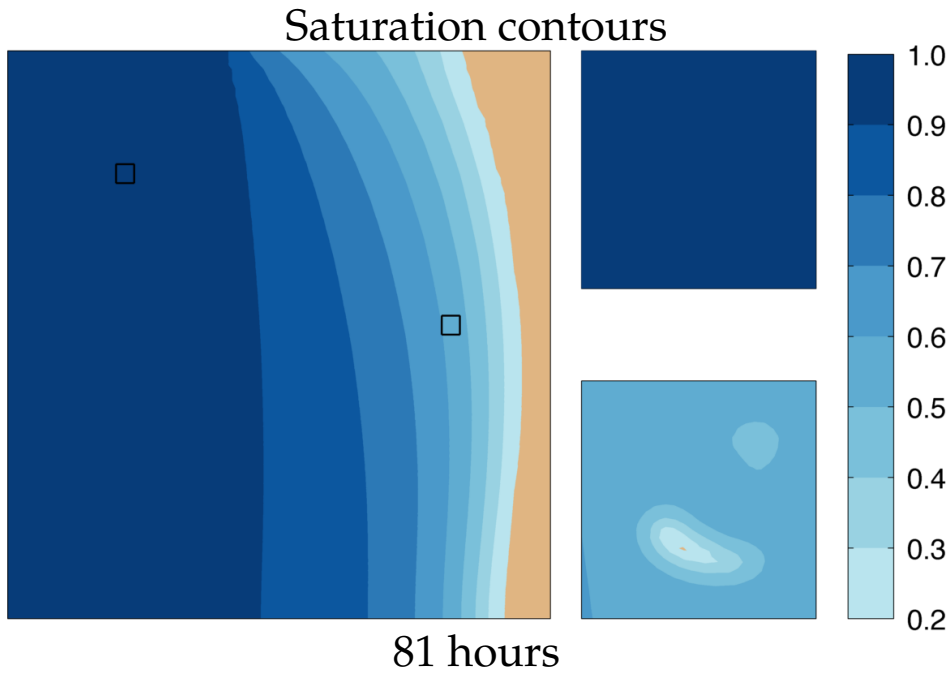
Simulation: Unsaturated water flow



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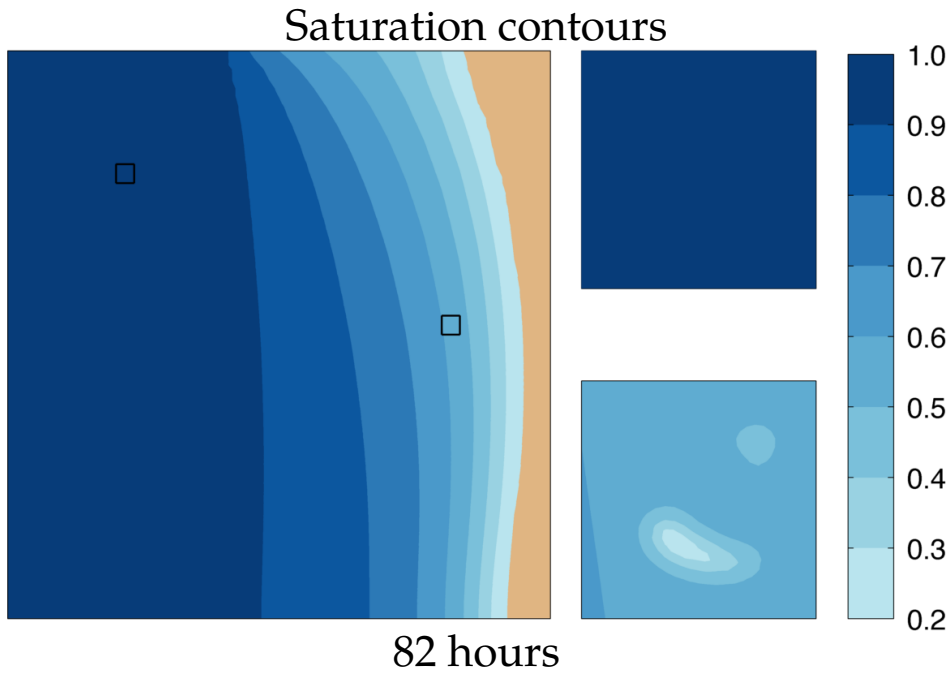
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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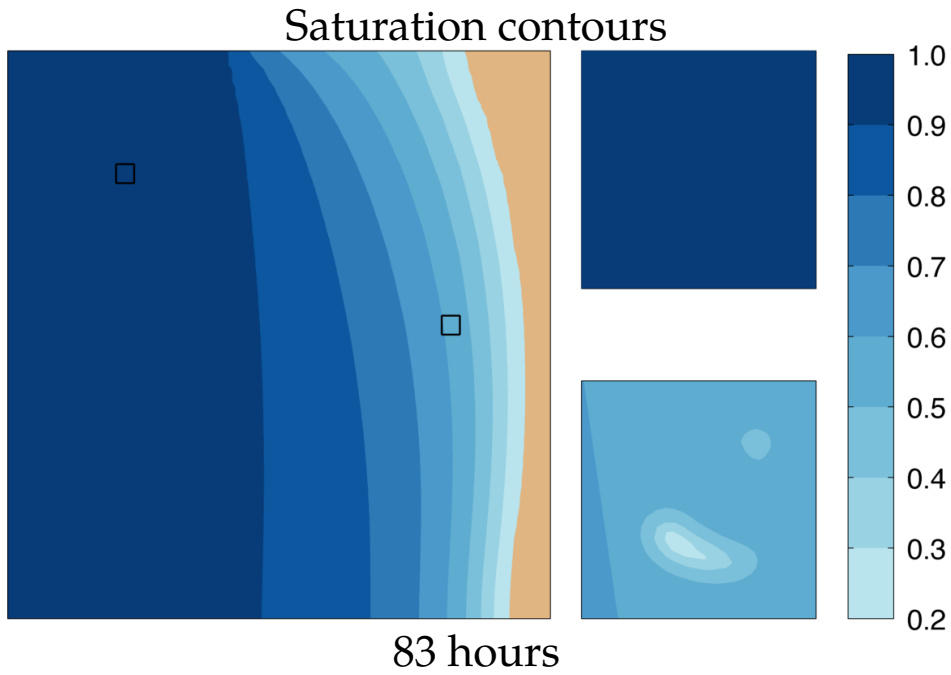
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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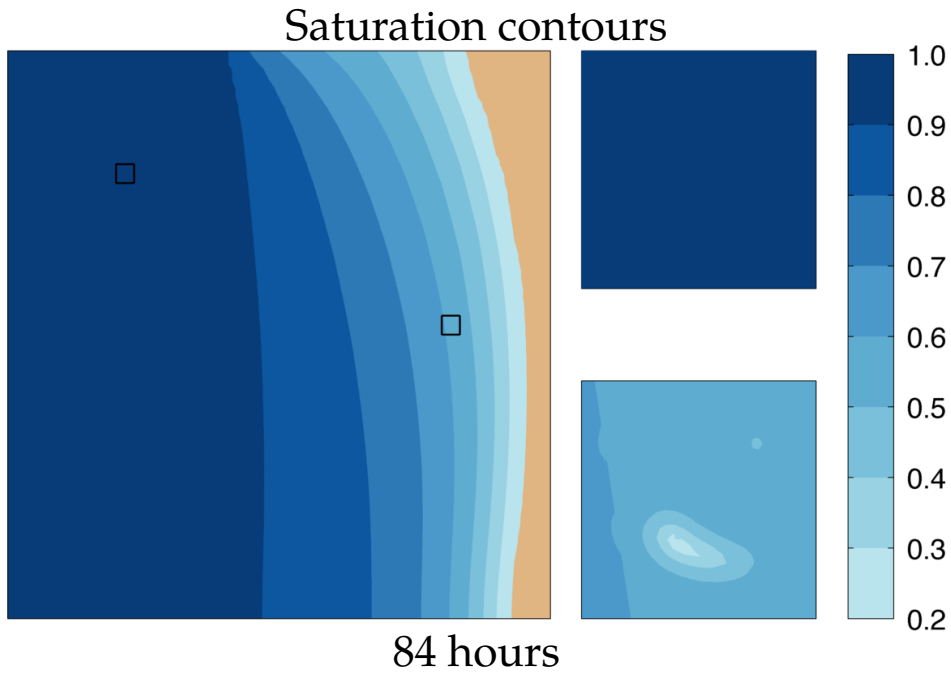
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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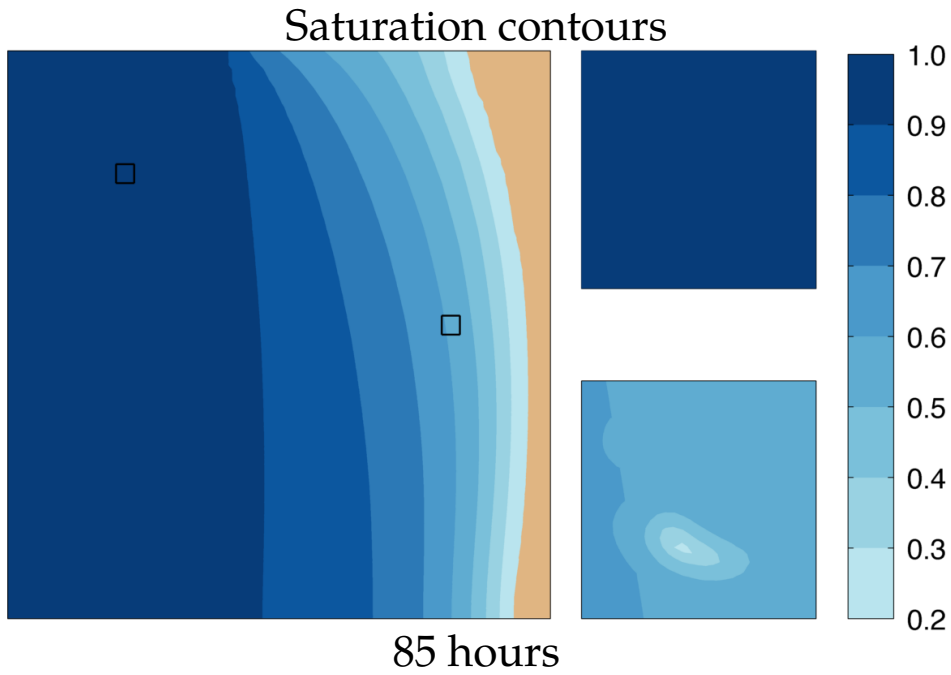
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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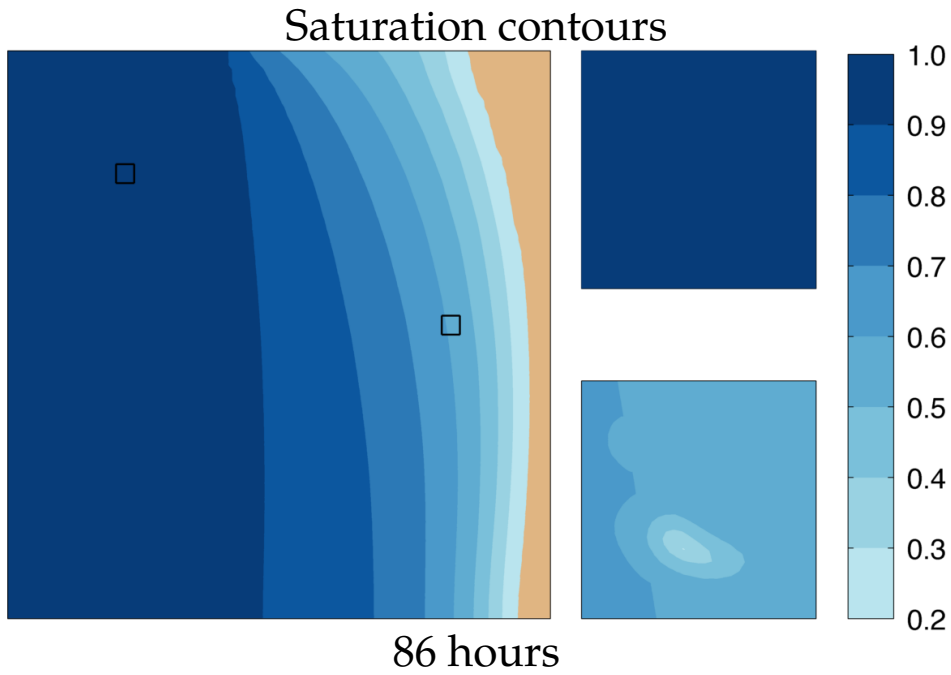
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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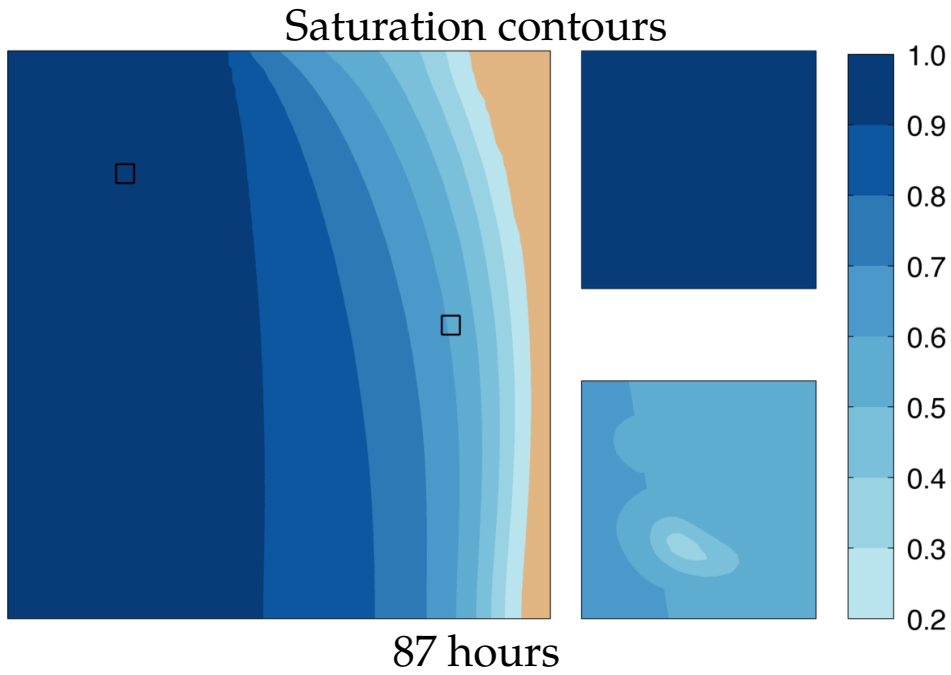
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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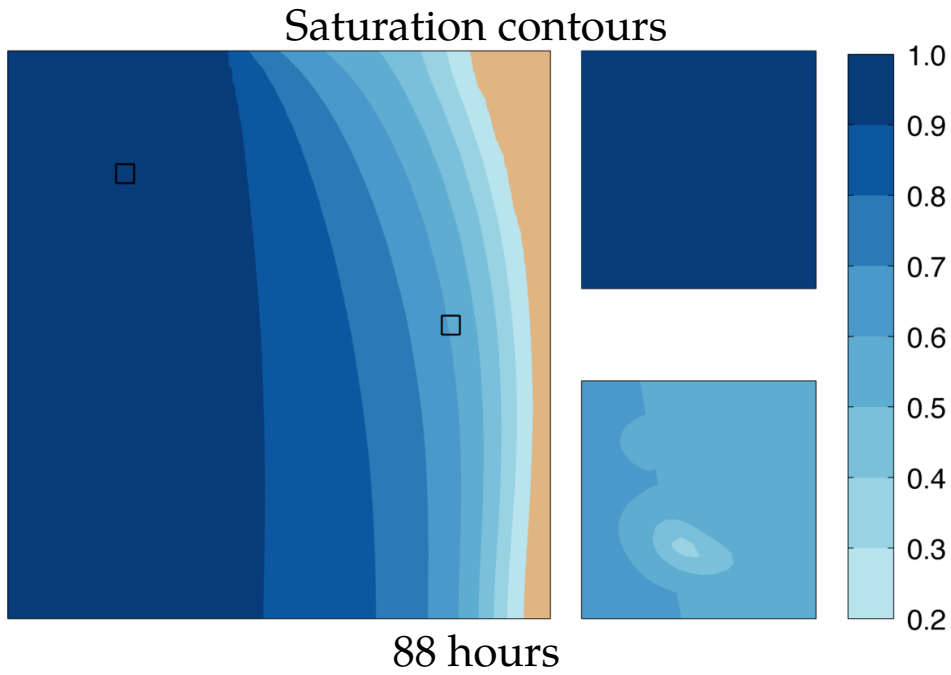
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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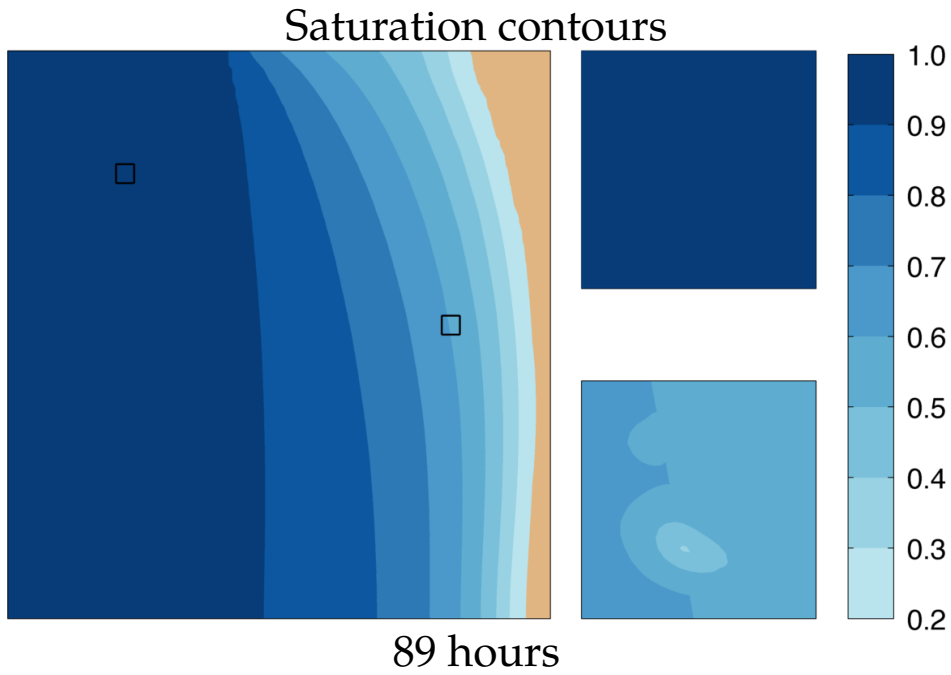
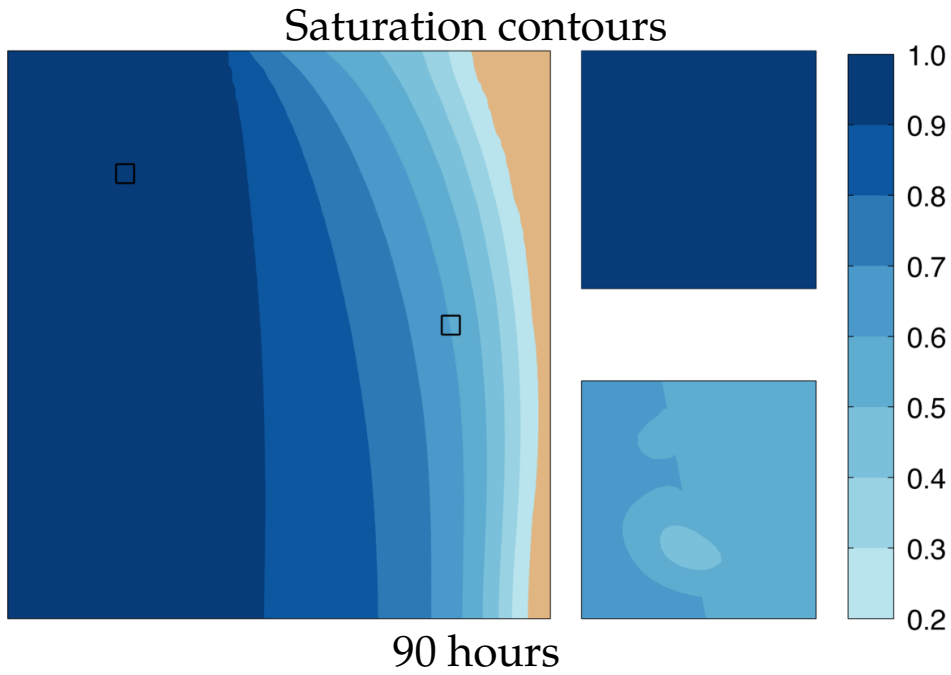


	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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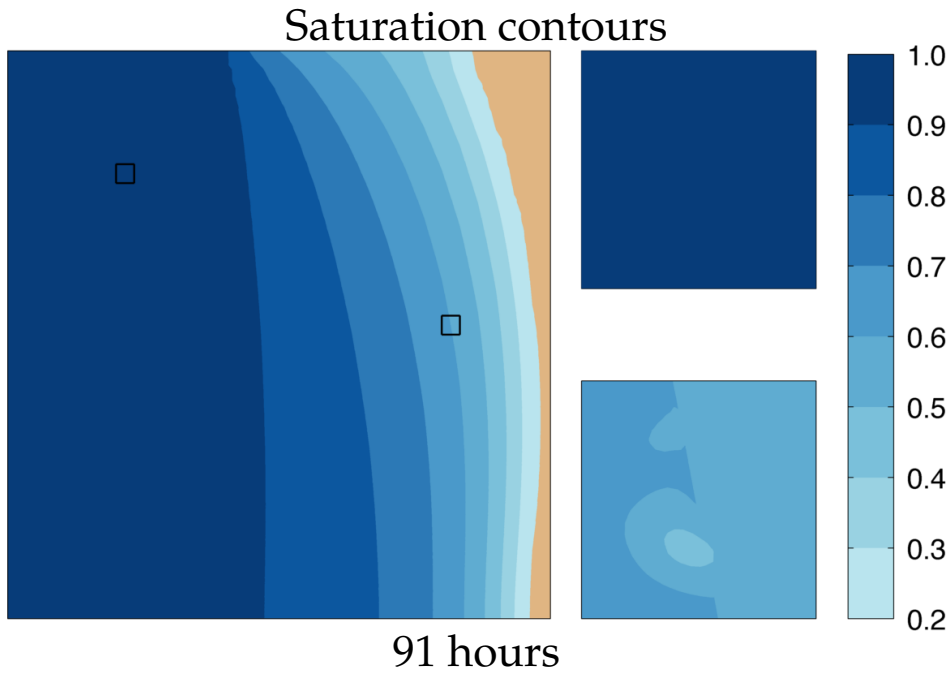
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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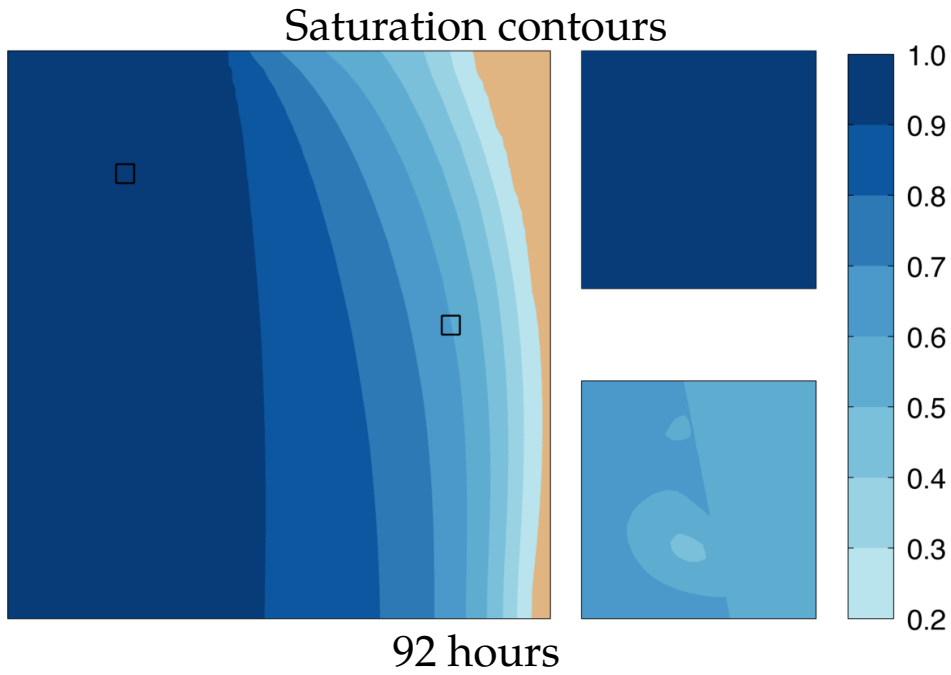
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
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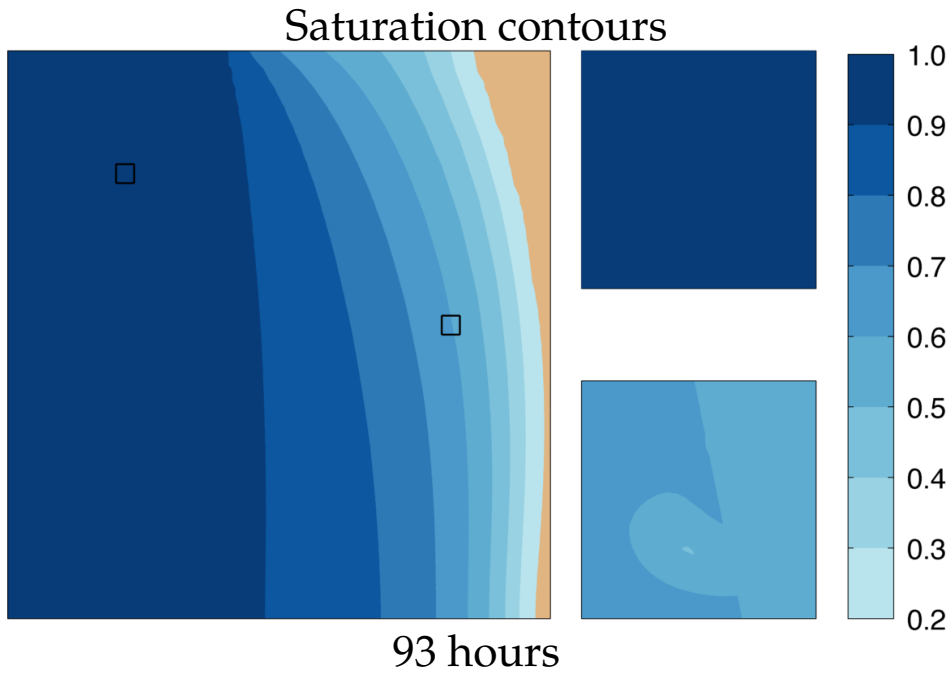
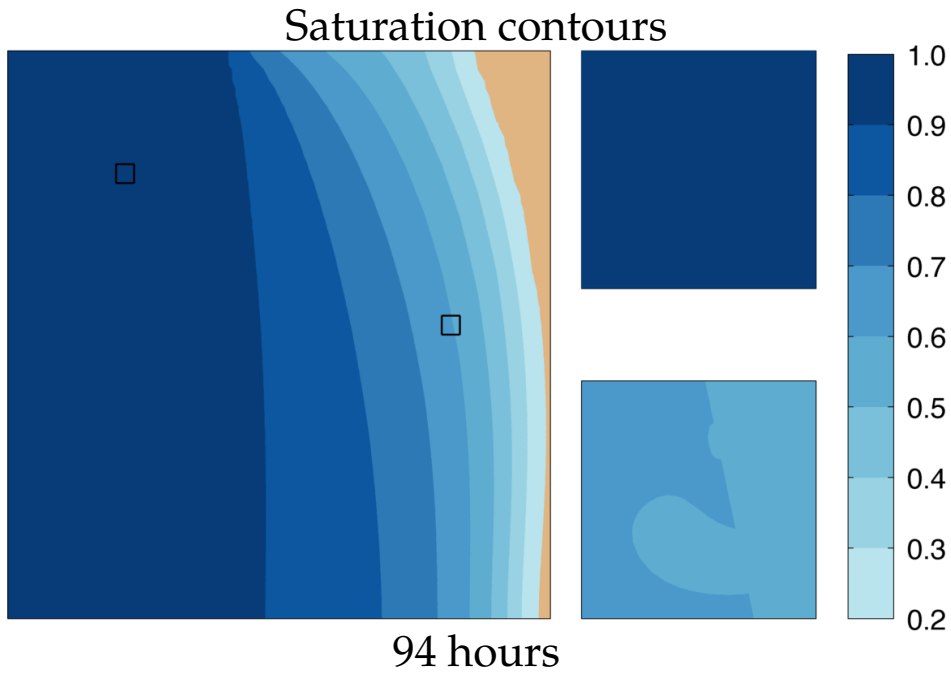
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
Speedup	1.00	0.98	1.90	3.50	5.84

Simulation: Unsaturated water flow



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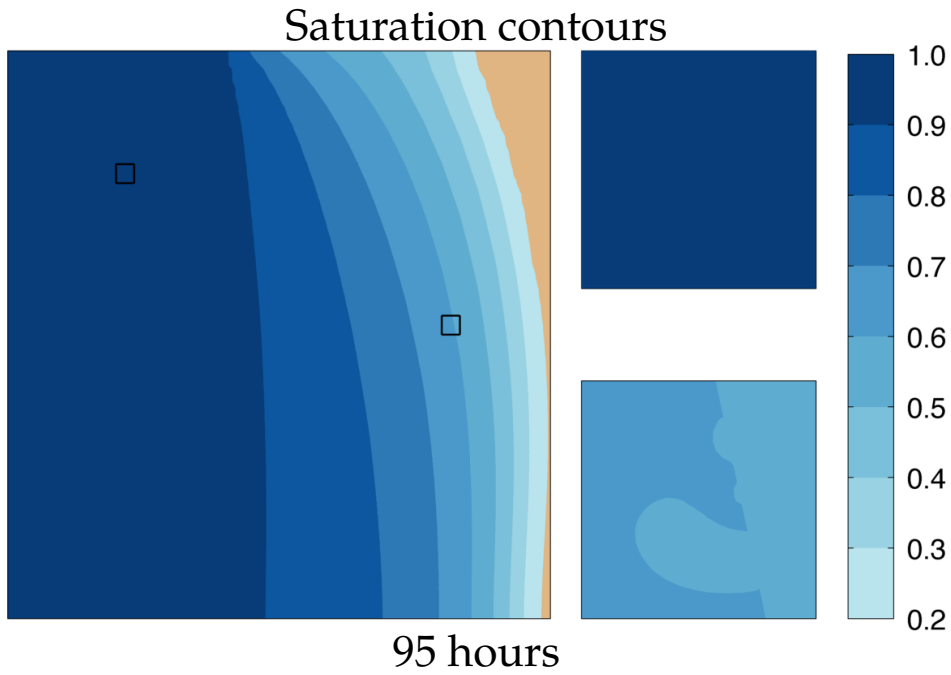

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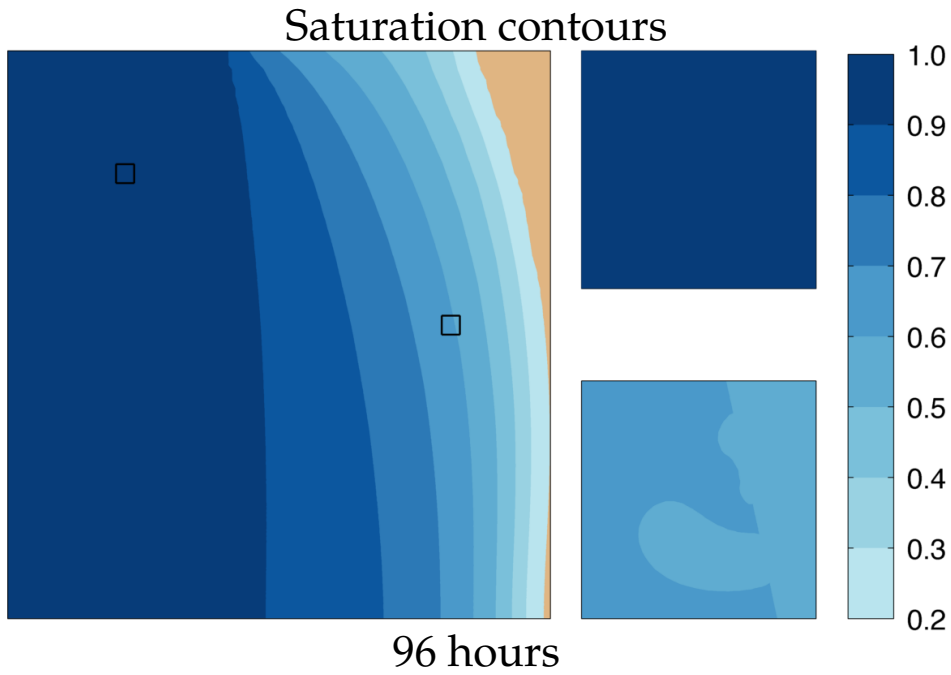

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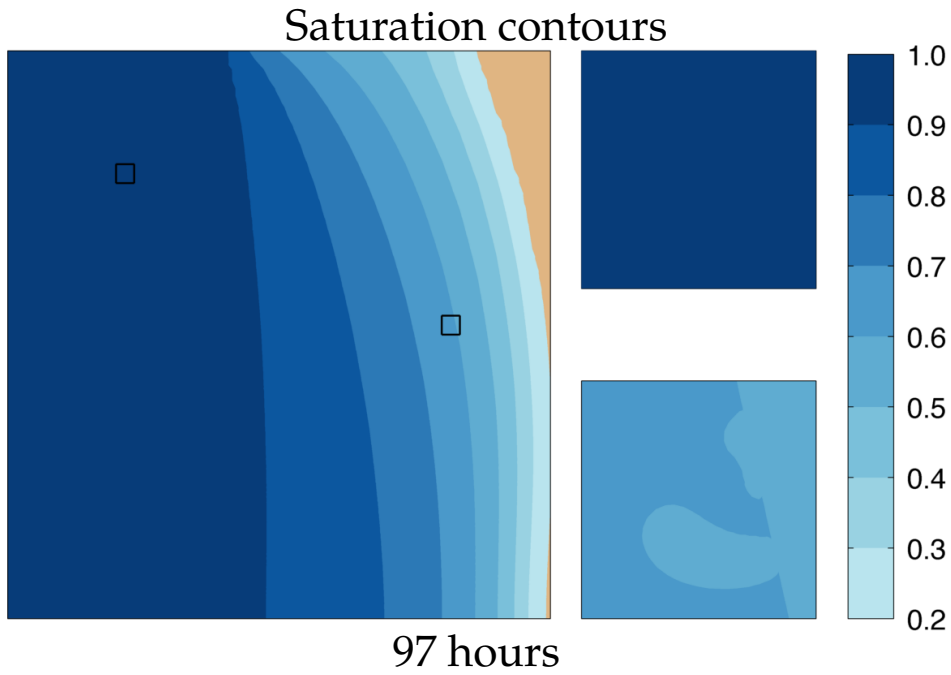

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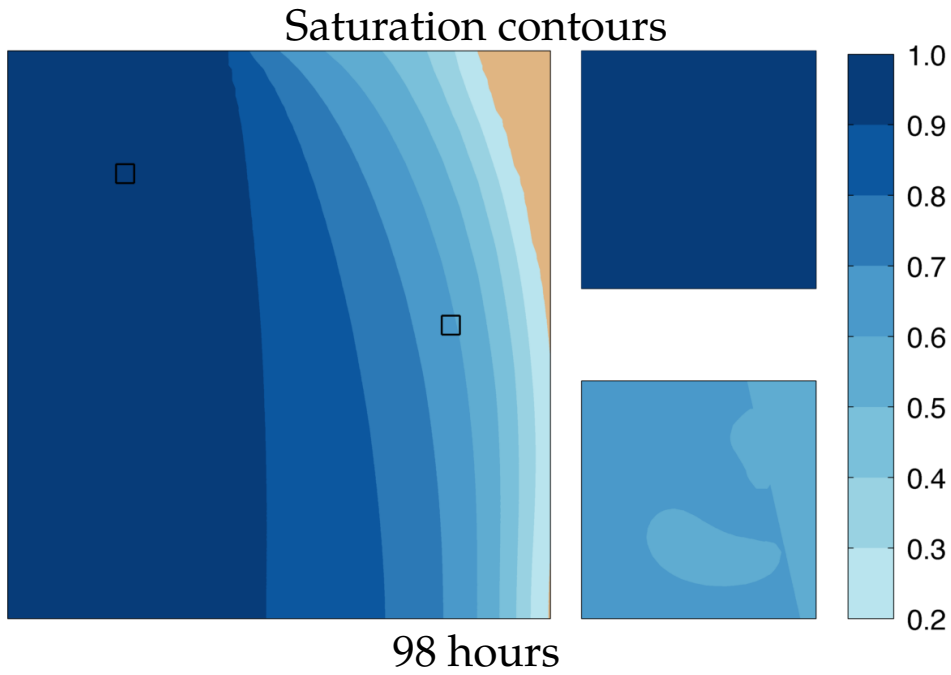

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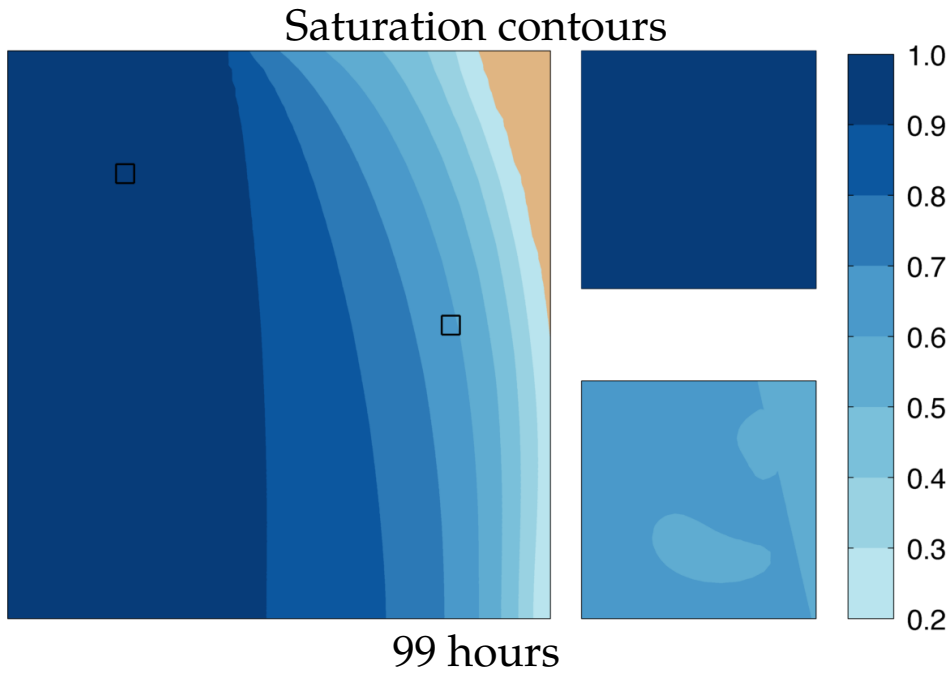

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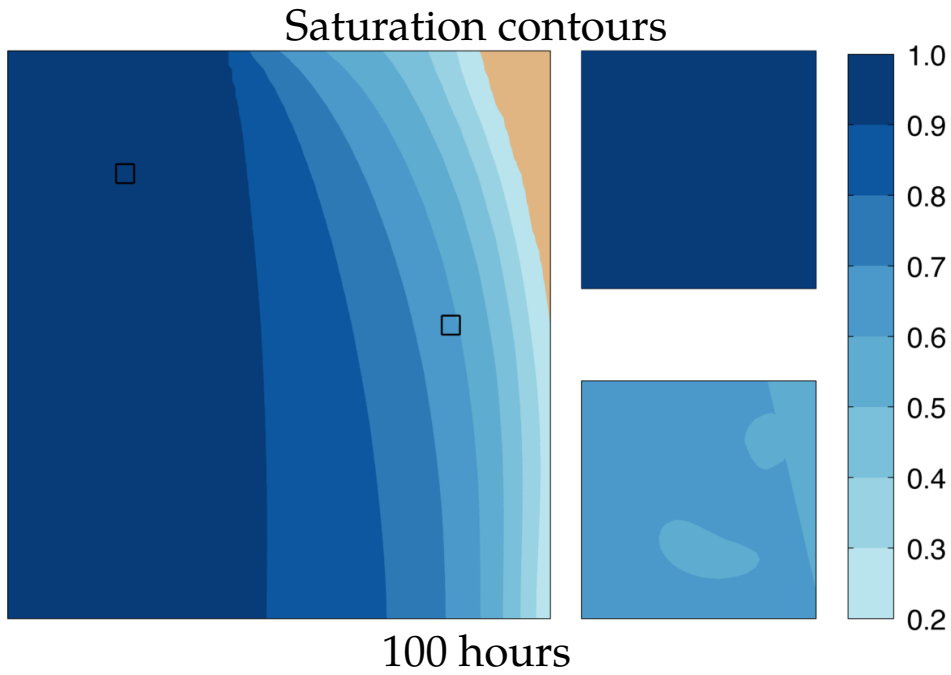
Simulation: Unsaturated water flow



	Serial	1 core	2 cores	4 cores	8 cores
Runtime (min)	179	182	94	51	31
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Summary and Conclusions

Pros and Cons of two-scale model:

- ✓ Ability to capture fine-scale detail in the full heterogeneous problem
- ✓ Produces good qualitative agreement with fine-scale model
- ✓ Significant reduction in simulation time (compared with fine-scale model)
- ✓ Numerically feasible for problems with very small-scale heterogeneities
- ✗ Restricted to heterogeneous domains comprised of two sub-domains where one is connected and the other forms disconnected/isolated inclusions

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Questions?

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