# Two-scale computational modelling of unsaturated flow in soils exhibiting small scale heterogeneity

Elliot J. Carr and Ian W. Turner
Mathematical Sciences, Queensland University of Technology
Presented at MPE 2013 (Mathematics of Planet Earth), Melbourne, Australia.

## Motivation

Unsaturated water flow in heterogeneous soils is commonly modelled using Richards' equation. In two spatial dimensions the equation can be written as

$$\frac{\partial}{\partial t}\big[\theta(h)\big] + \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \big[-K(h)\nabla_{\boldsymbol{x}}(h + x_2)\big] = 0\,,$$

where $h$ is the capillary pressure head [L], $K$ is the hydraulic conductivity [$\mathrm{L\,T^{-1}}$], $\theta$ is the volumetric moisture content [−] and $\boldsymbol{x} = (x_1, x_2)$ is the coordinate variable. A problem arises, however, when the scale of the soil heterogeneities is small. In this case, numerical solution of the model is computationally intractable due to the excessive mesh refinement required to represent the actual geometry and the severe discontinuity in the hydraulic properties (e.g. conductivity) across soil interfaces.

## Two-scale model for a two soil medium

Assume that at each point in the macroscopic domain ($\boldsymbol{x} \in \Omega$), there exists a cell $Y$ whose geometry is representative of the material heterogeneity at the point $\boldsymbol{x}$ (see Figure 1). The cell $Y$ is comprised of two sub-domains $Y_a$ and $Y_b$ that are occupied by soil $a$ and soil $b$ respectively. Soil $a$ is macroscopically connected and assumed to have high conductivity compared to soil $b$ (see paper [2] for full list of hydraulic properties).

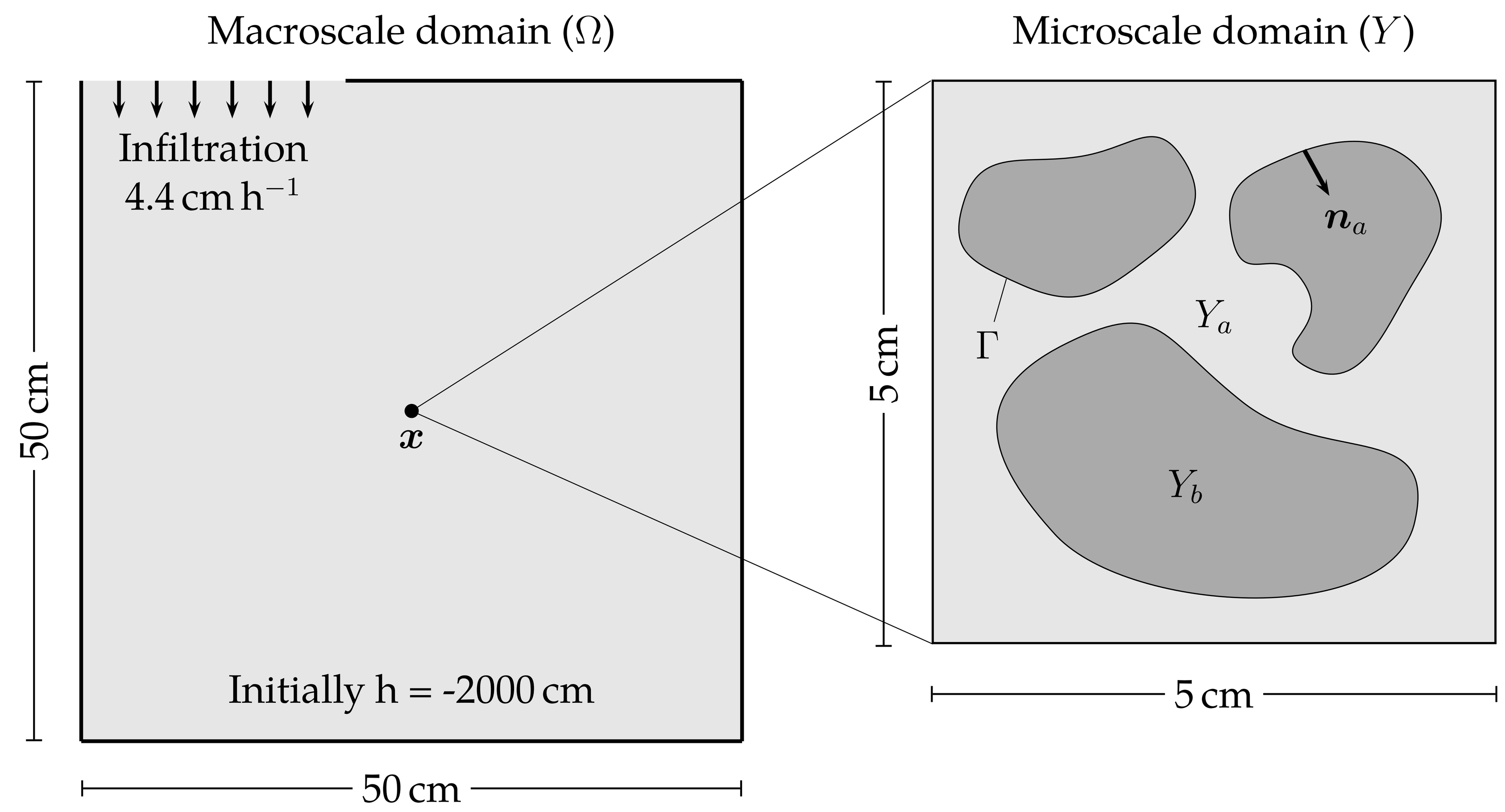


**Figure 1:** Two-scale domain configuration.

The two-scale model of Szymkiewicz and Lewandowska [2] consists of an upscaled equation for the macroscopic flow:

$$\frac{\partial}{\partial t}\big[\varepsilon_a\theta_a(h_a)\big] + \boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \big[-\boldsymbol{K}_{\text{eff}}(h_a)\nabla_{\boldsymbol{x}}\big(h_a + x_2\big)\big] = Q\,, \qquad \boldsymbol{x} \in \Omega\,,$$

where $\varepsilon_a = |Y_a|/|Y|$, coupled with a microscopic equation at each $\boldsymbol{x} \in \Omega$ governing the flow in $Y_b$:

$$\frac{\partial}{\partial t}\big[\theta_b(h_b)\big] + \boldsymbol{\nabla}_{\boldsymbol{y}} \cdot \big[-K_b(h_b)\nabla_{\boldsymbol{y}}h_b\big] = 0\,, \qquad \boldsymbol{y} \in Y_b\,.$$

Equality of the pressure head values is imposed at the soil interface:

$$h_b(\boldsymbol{y}, t) = h_a(\boldsymbol{x}, t)\,, \qquad \boldsymbol{y} \in \Gamma\,.$$

The source term $Q$ equals the amount of fluid flux across the interface (from $Y_b$ to $Y_a$) scaled by the area of $Y$:

$$Q = \frac{1}{|Y|}\int_{\Gamma} K_b(h_b)\nabla_{\boldsymbol{y}}h_b \cdot \boldsymbol{n}_a\, ds\,.$$

The effective conductivity $\boldsymbol{K}_{\text{eff}}$ is defined in terms of the solution of a periodic problem on $Y$ according to the homogenization theory [2]. Note that the two-scale model features coupling in both directions: the macroscopic variable $h_a$ appears in the boundary condition for the microscopic equation and the macroscopic equation features the source term $Q$ calculated from the microscopic variable $h_b$.

## Numerical solution approach

We discretise in space using the control volume finite element method and assume that an individual unit cell $Y$ is associated with each macroscopic node. We use a structured grid at the macroscale (1681 nodes) and an unstructured grid at the microscale (222 nodes) giving a total of 374,863 unknowns. A novel aspect of our approach is the use of an exponential integrator to resolve both scales in a completely coupled manner. We employ the exponential Rosenbrock-Euler method using the implementation proposed by Carr et al. [1]. Each two-scale simulation takes roughly 2 hours to run in Matlab.

## Results

Pressure head fields after 25 hours of infiltration are given below for the conductivity ratios: $K_b/K_a = 10^{-2}$ (left) and $10^{-6}$ (right).

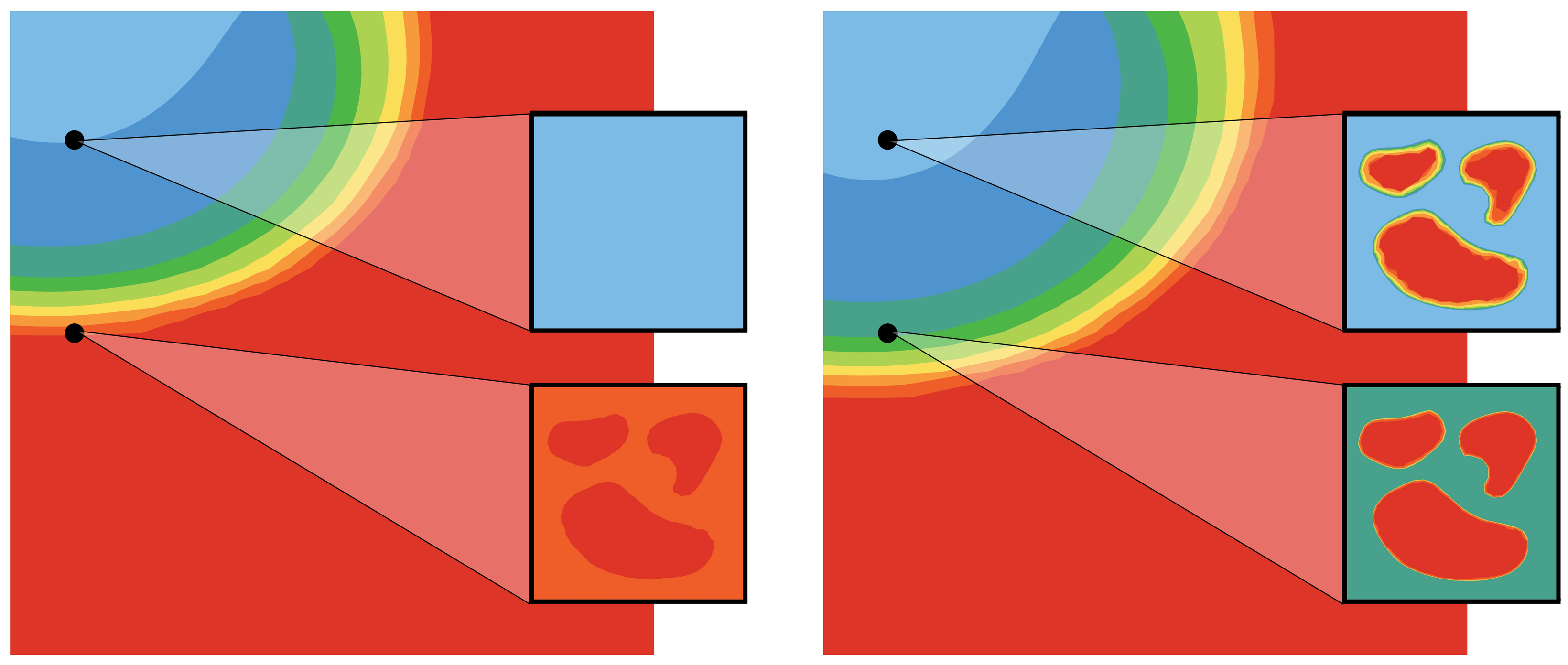

**Figure 2:** Two-scale model.

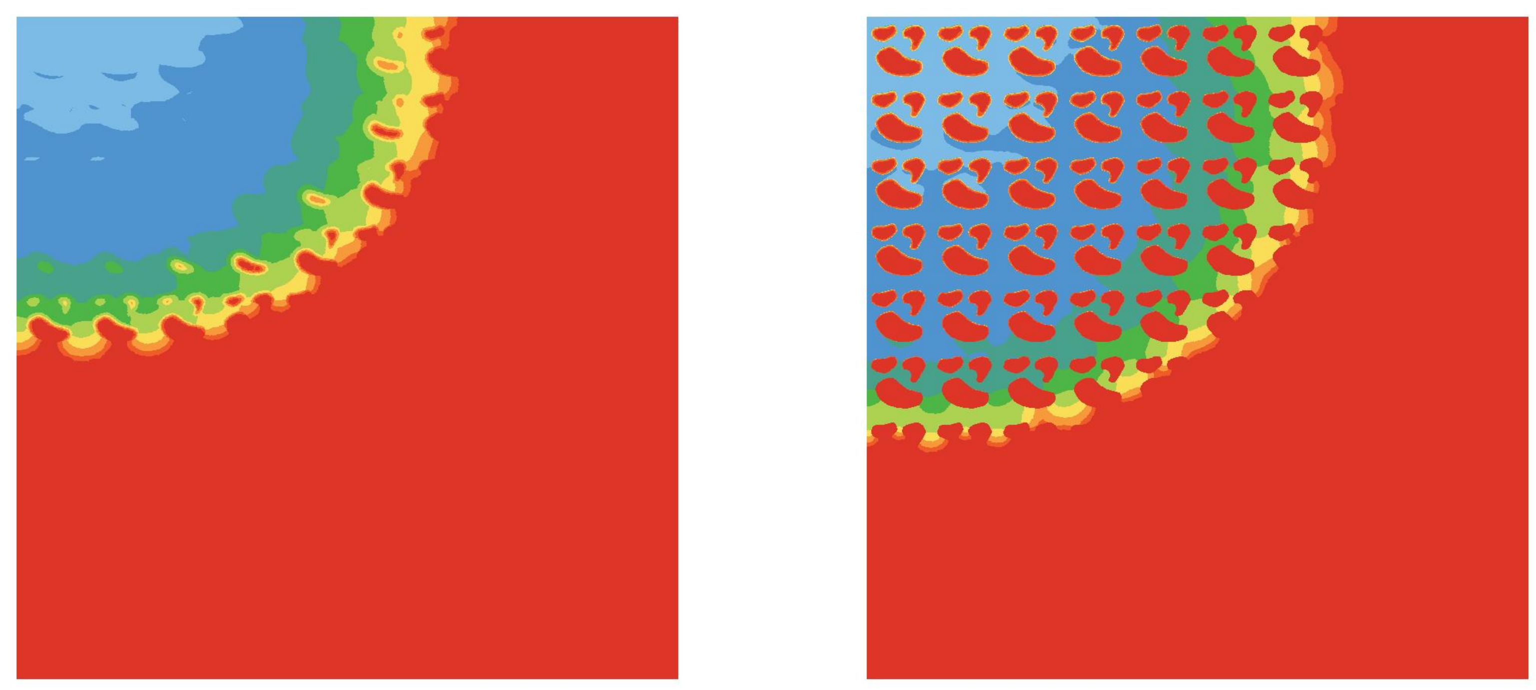

**Figure 3:** Fine-scale model.

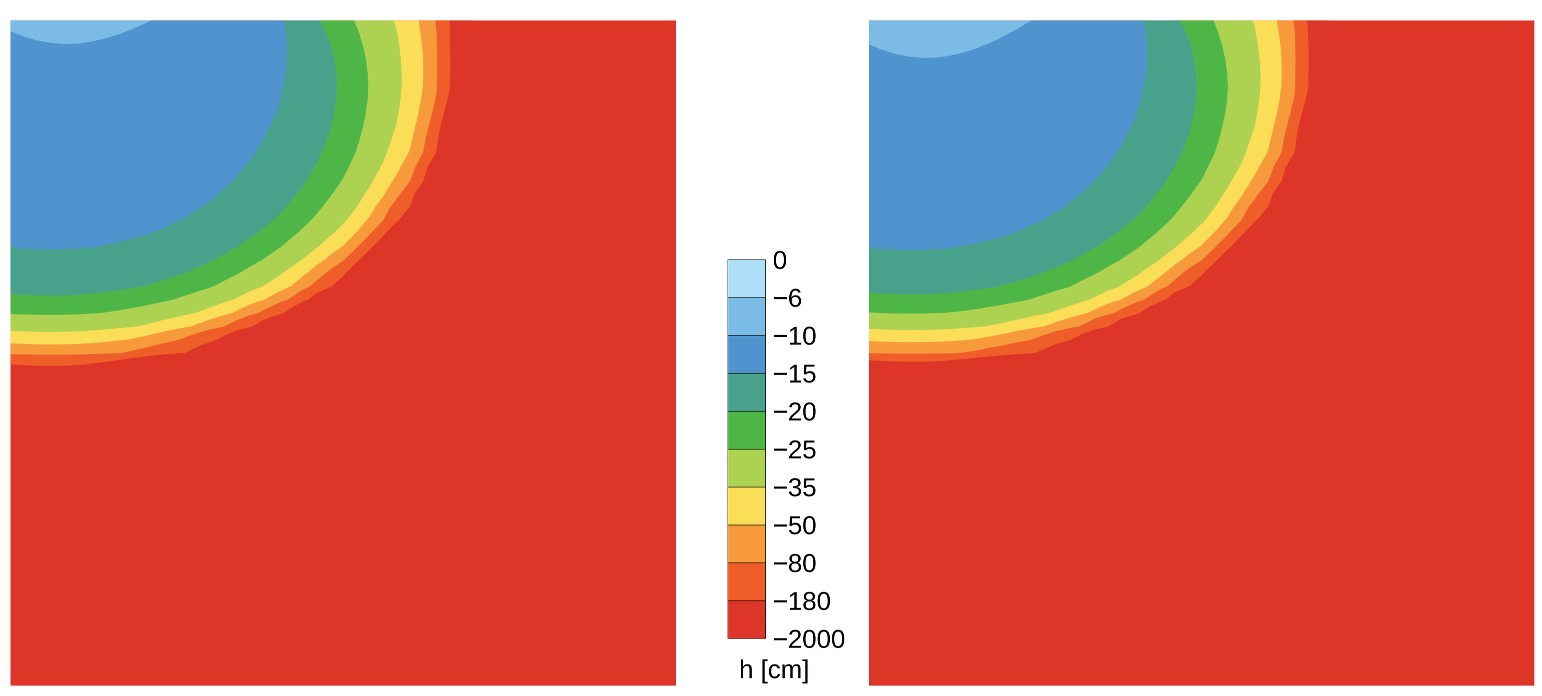


**Figure 4:** Macroscopic model.

The two-scale model predicts the faster propagation of the wetting front for smaller values of the conductivity ratio. A classical macroscopic (upscaled) model featuring Richards' equation with effective/homogenized parameters is unable to capture this behaviour.

## References

[1] E. J. Carr, T. J. Moroney, and I. W. Turner. Efficient simulation of unsaturated flow using exponential time integration. *Appl. Math. Comput.*, 217(14):6587–6596, 2011.

[2] A. Szymkiewicz and J. Lewandowska. Micromechanical approach to unsaturated water flow in structured geomaterials by two-scale computations. *Acta Geotechnica*, 3:37–47, 2008.