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Novel calculation of response times for groundwater flow

Dr Elliot Carr
Joint work with Professor Matthew Simpson (QUT)

elliot.carr@qut.edu.au @ElliotJCarr https://elliotcarr.github.io/

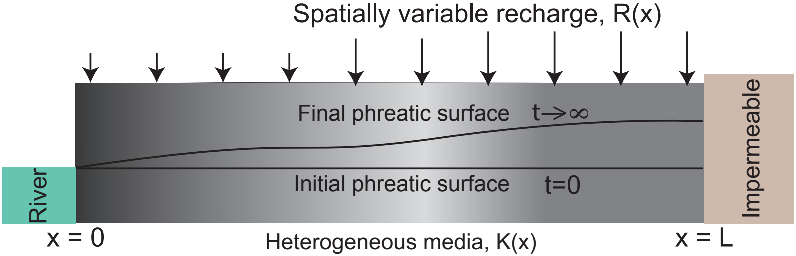

Spatially variable recharge, R(x)
Final phreatic surface t→∞
Initial phreatic surface t=0
River
Impermeable
x = 0
x = L
Heterogeneous media, K(x)

Figure 1: **?**

- One-dimensional linearised Dupuit-Forchheimer model of saturated flow [**?**]:

$$S\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(K(x)\bar{h}\frac{\partial h}{\partial x}\right) + R(x), \quad 0 < x < L, \;\; t > 0,$$

$$h(x,0) = h_0(x), \quad h(0,t) = h_1, \quad \frac{\partial h}{\partial x}(L,t) = 0.$$

- Nomenclature:

$$\begin{aligned} h(x,t)&: \quad \text{saturated thickness [L]}\\ K(x)&: \quad \text{saturated hydraulic conductivity [LT}^{-1}\text{]}\\ R(x)&: \quad \text{recharge rate [LT}^{-1}\text{]}\\ S&: \quad \text{storage coefficient [}-\text{]} \end{aligned}$$

- Definition of response time, t_r:

$$\frac{h(x,t_r) - h_\infty(x)}{h_0(x) - h_\infty(x)} = \delta,$$

where δ is a small prescribed tolerance (e.g. $\delta = 10^{-p}$).

Response time approximations

Concept of mean action time

- Transition can be represented as a CDF:

$$F(t;x) := 1 - \frac{h(x,t) - h_\infty(x)}{h_0(x) - h_\infty(x)}.$$

- **Mean action time (MAT)**: common measure of the time required to reach steady state []:

$$\mathrm{MAT}(x) := \int_0^\infty t\,f(t;x)\,\mathrm{d}t; \quad f(t;x) = \frac{\partial F}{\partial t}(t;x).$$

- **Variance of action time (VAT)** [m (i)]:

$$\mathrm{VAT}(x) := \int_0^\infty t^2 f(t;x)\,\mathrm{d}t - \left[\int_0^\infty t\,f(t;x)\,\mathrm{d}t\right]^2.$$

- Response time approximations [m (i)]:

$$t_r \approx \mathrm{MAT}(L), \quad t_r \approx \mathrm{MAT}(L) + \sqrt{\mathrm{VAT}(L)}.$$

Computing the moments
Moments satisfy boundary value problems

- Definition of kth raw moment:

$$M_k(x) = \int_0^\infty t^k f(t;x)\,\mathrm{d}t,$$

$$f(t;x) = \frac{1}{h_\infty(x) - h_0(x)}\frac{\partial}{\partial t}\left[h(x,t) - h_\infty(x)\right].$$

- Boundary value problem for scaled moment $\overline{M}_k(x) = M_k(x)(h_\infty(x) - h_0(x))$ [**?**]:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(D(x)\frac{\mathrm{d}\overline{M}_k}{\mathrm{d}x}\right) = -k\overline{M}_{k-1}(x), \quad 0 < x < L,$$

$$\overline{M}_k(0) = 0, \quad \frac{\mathrm{d}\overline{M}_k}{\mathrm{d}x}(L) = 0.$$

  where $D(x) = \overline{h}K(x)/S$.

- Recursively solve boundary value problems. Starting with $k = 1$ and given $\overline{M}_0(x) = h_\infty(x) - h_0(x)$. Repeat until a desired order.

Calculating the release time

An asymptotic estimate based on the moments

- At each location x, transient solution takes the form:

$$h(x,t) = h_\infty(x) + \sum_{n=1}^{N} c_n e^{-t\beta_n}.$$

- Follows that:

$$\frac{h(x,t) - h_\infty(x)}{h_0(x) - h_\infty(x)} = \sum_{n=1}^{N} \alpha_n e^{-t\beta_n} \simeq \alpha_1 e^{-t\beta_1}, \quad \text{for large } t.$$

- Asymptotic estimate of response time:

$$\frac{h(x,t_r) - h_\infty(x)}{h_0(x) - h_\infty(x)} = \delta \quad \Rightarrow \quad \alpha_1 e^{-t_r\beta_1} \simeq \delta \quad \Rightarrow \quad t_r \simeq \frac{1}{\beta_1}\log\left(\frac{\alpha_1}{\delta}\right),$$

where δ is a small prescribed tolerance (e.g. $\delta = 10^{-p}$).

Calculating the release time
An asymptotic estimate based on the moments

- Asymptotic relation:

$$M_k(x) \simeq k!\,\frac{\alpha_1}{\beta_1^k} \quad \text{for large } k.$$

- Matching consecutive moments:

$$\begin{aligned} (k-1)!\,\frac{\alpha_1}{\beta_1^{k-1}} &\simeq M_{k-1}(x) \\ k!\,\frac{\alpha_1}{\beta_1^k} &\simeq M_k(x) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \alpha_1 &\simeq \frac{M_k(x)}{k!}\left(\frac{kM_{k-1}(x)}{M_k(x)}\right)^k \\ \beta_1 &\simeq \frac{kM_{k-1}(x)}{M_k(x)}. \end{aligned}$$

- Asymptotic estimate of release time:

$$t_r \simeq \frac{M_k(x)}{kM_{k-1}(x)} \log\left[\frac{M_k(x)}{k!\,\delta}\left(\frac{kM_{k-1}(x)}{M_k(x)}\right)^k\right] \quad \text{for large } k.$$

- No need for transient solution $c_i(r,t)$!

- Consider the homogeneous version of the model:

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial h}{\partial x}\right) + W, \quad 0 < x < L, \ \ t > 0,$$

$$h(x,0) = h_1, \quad h(0,t) = h_1, \quad \frac{\partial h}{\partial x}(L,t) = 0.$$

where $D = hK/S$ and $W = R/S$.

- At $x = L$, the first four moments are given by:

$$M_0(L) = \frac{1}{1}\frac{L^0}{D^0}, \quad M_1(L) = \frac{5}{12}\frac{L^2}{D}, \quad M_2(L) = \frac{61}{180}\frac{L^4}{D^2}, \quad M_3(L) = \frac{1385}{3360}\frac{L^6}{D^3}.$$

- In general, we have:

$$M_k(L) = \frac{E_{k+2}}{(2k+2)!/(2k!)}\frac{L^{2k}}{D^k},$$

where E_i denotes the ith Euler number (https://oeis.org/A000364).

- Recall, the asymptotic estimate of release time:

$$t_r \simeq \frac{M_k(x)}{kM_{k-1}(x)} \log\left[\frac{M_k(x)}{k!\,\delta}\left(\frac{kM_{k-1}(x)}{M_k(x)}\right)^k\right] \quad \text{for large } k.$$

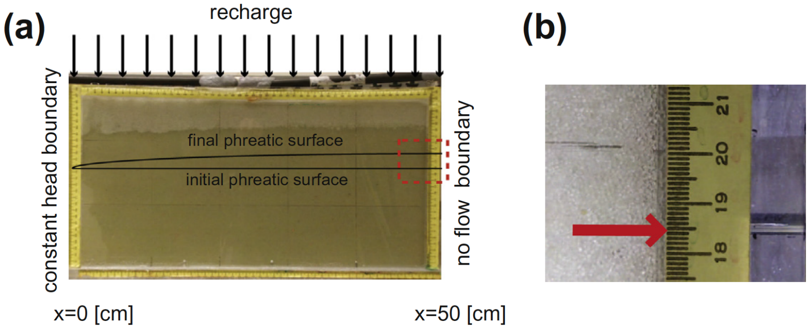
- For large k, there is an approximation for the Euler numbers [?]

$$M_k(L) = \frac{E_{k+2}}{(2k+2)!/(2k!)}\frac{L^{2k}}{D^k} \simeq \frac{4^{k+2}(2k+2)!}{(2k+2)!/(2k!)\pi^{2k+3}}\frac{L^{2k}}{D^k} \simeq \frac{k!}{2}\frac{4^{k+2}}{\pi^{2k+3}}\frac{L^{2k}}{D^k} \quad \text{for large } k.$$

- Asymptotic estimate of release time:

$$t_r \simeq \frac{4}{\pi^2}\frac{L^2}{D}\log_e\left(\frac{32}{\pi^3\delta}\right).$$

- This is precisely the value of t_r that would be obtained from taking the first term in the Fourier series solution.



(a)
recharge
constant head boundary
final phreatic surface
initial phreatic surface
no flow boundary
x=0 [cm]
x=50 [cm]
(b)
21
20
19
18

Figure 2: m (i)

Results for Homogeneous flow

Comparison to experimental data

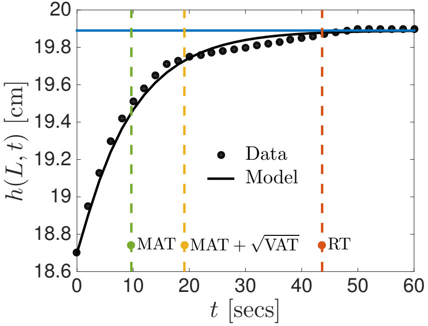


Figure 3: **?** [$\delta = 10^{-2}$]

Results for Homogeneous flow

How many moments are required?

k	t_r	δ_r	$\lvert\delta_r - \delta\rvert$
1	44.5556	0.01	9.3e-04
2	43.7157	0.01	8.5e-05
3	43.6410	0.01	6.0e-06
4	43.6356	0.01	2.3e-07
5	43.6353	0.01	2.4e-08
6	43.6354	0.01	8.3e-09
7	43.6354	0.01	1.5e-09
8	43.6354	0.01	2.4e-10
9	43.6354	0.01	3.4e-11
10	43.6354	0.01	4.7e-12

Table 1: **?** [$\delta = 10^{-2}$]

$$\delta_r = \frac{h(L,t_r) - h_\infty(L)}{h_0(L) - h_\infty(L)}; \quad t_r \simeq \frac{M_k(x)}{kM_{k-1}(x)} \log\left[\frac{M_k(x)}{k!\,\delta}\left(\frac{kM_{k-1}(x)}{M_k(x)}\right)^k\right].$$

$$K(x) = 81.0707 + 64\left[e^{-0.1(x-50/3)^2} - e^{-0.1(x-100/3)^2}\right]$$

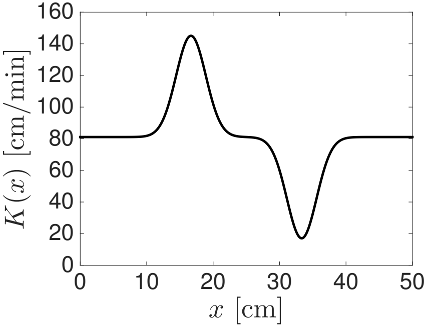


Figure 4: **?**

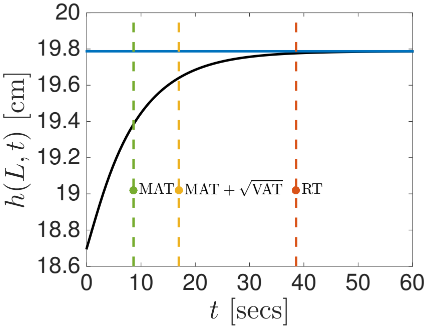


Figure 5: **?** [$\delta = 10^{-2}$]

k	t_r	δ_r	$\lvert\delta_r - \delta\rvert$
1	39.7338	0.01	1.3e-03
2	38.7041	0.01	1.6e-04
3	38.5858	0.01	1.5e-05
4	38.5743	0.01	7.2e-07
5	38.5736	0.01	1.3e-07
6	38.5737	0.01	5.6e-08
7	38.5737	0.01	1.4e-08
8	38.5737	0.01	3.0e-09
9	38.5737	0.01	6.0e-10
10	38.5737	0.01	1.1e-10

Table 2: **?** [$\delta = 10^{-2}$]

$$\delta_r = \frac{h(L,t_r) - h_\infty(L)}{h_0(L) - h_\infty(L)}; \quad t_r \simeq \frac{M_k(x)}{kM_{k-1}(x)} \log\left[\frac{M_k(x)}{k!\,\delta}\left(\frac{kM_{k-1}(x)}{M_k(x)}\right)^k\right].$$

Conclusions and Summary

Summary

- Extended the mean action time concept.
- New method for calculating response times using higher-order moments.
- New estimate is significantly more accurate than existing estimates based on low-order moments.

Extensions

- Techniques presented carry over to two and three dimensional problems.
- Nonlinear problems?

Further reading

Carr and Simpson (2018)

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Elliot J. Carr *, Matthew J. Simpson

School of Mathematical Sciences, Queensland University of Technology (QUT), Brisbane, Australia

A R T I C L E   I N F O

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A B S T R A C T

We study measures of the amount of time required for transient flow in heterogeneous porous media to effectively reach steady state, also known as the response time. Here, we develop a new approach that extends the concept of mean action time. Previous applications of the theory of mean action time to estimate the response time use the first two central moments of the probability density function associated with the transition from the initial condition, at $t = 0$, to the steady state condition that arises in the long time limit, as $t \to \infty$. This previous approach leads to a computationally convenient estimation of the response time, but the accuracy can be poor. Here, we outline a powerful extension using the first k raw moments, showing how to produce an extremely accurate estimate by making use of asymptotic properties of the cumulative distribution function. Results are validated using an existing laboratory-scale data set describing flow in a homogeneous porous medium. In addition, we demonstrate how the results also apply to flow in heterogeneous porous media. Overall, the new method is: (i) extremely accurate; and (ii) computationally inexpensive. In fact, the computational cost of the new method is orders of magnitude less than the computational effort required to study the response time by solving the transient flow equation. Furthermore, the approach provides a rigorous mathematical connection with the heuristic argument that the response time for flow in a homogeneous porous medium is proportional to L^2/D, where L is a relevant length scale, and D is the aquifer diffusivity. Here, we extend such heuristic arguments by providing a clear mathematical definition of the proportionality constant.

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