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Perturbation solution for heat/mass transfer across an irregular interface

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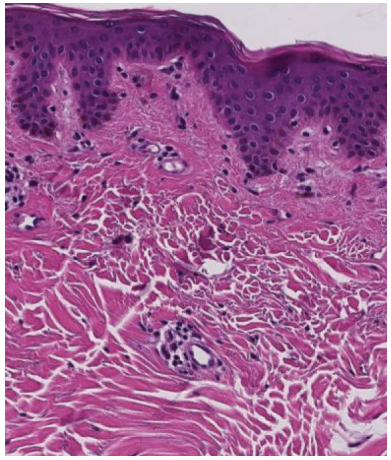
Diffusion across an irregular interface

Motivating application

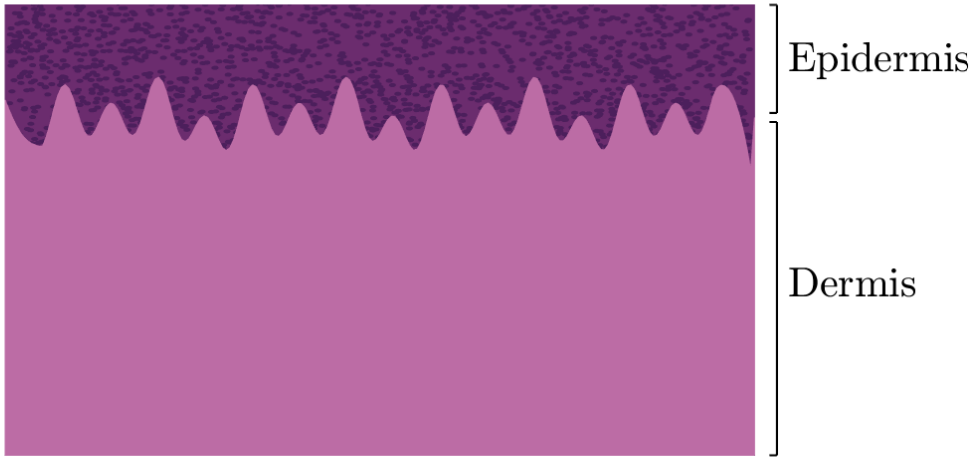
Treatment of accidental scald burn injuries.

First aid treatment: Immediate application of cool running water.

Mathematical modelling used to inform duration and temperature of cool water treatment [Andrews et al. (2016); McInerney et al. (2019); Simpson et al. (2017)].



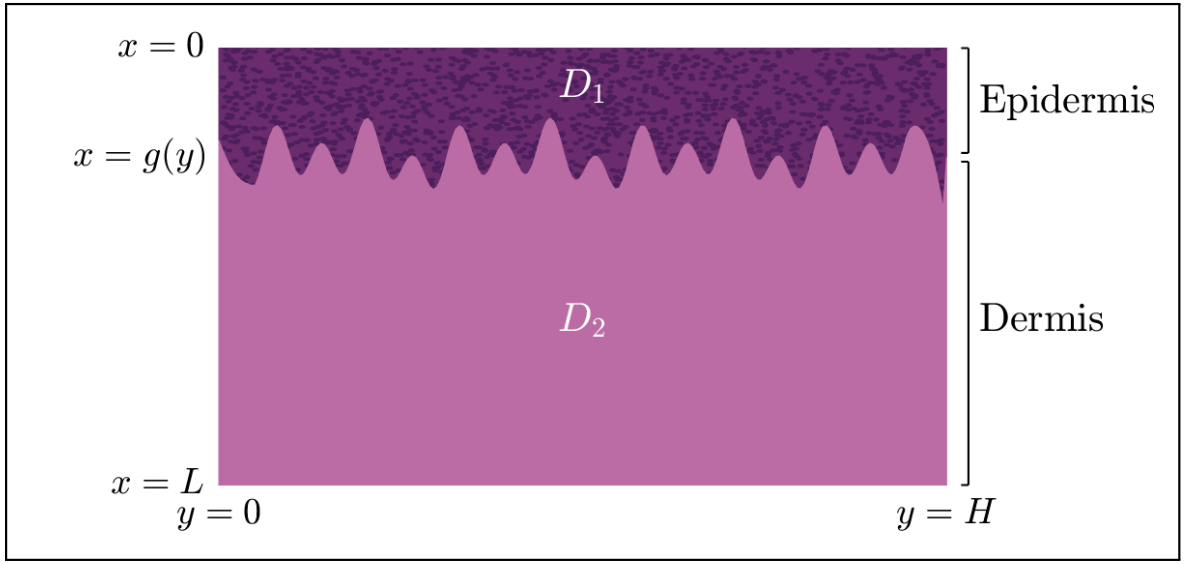
Haridas et al. (2017)



Schematic

Diffusion across an irregular interface

Governing equations



$$\frac{\partial u_1}{\partial t} = D_1 \Delta u_1, \quad 0 < x < g(y),$$

$$\frac{\partial u_2}{\partial t} = D_2 \Delta u_2, \quad g(y) < x < L,$$

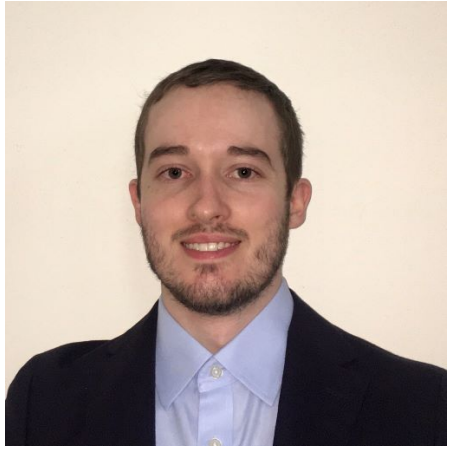
$$u_1(g(y), y, t) = u_2(g(y), y, t),$$

$$D_1 \nabla u_1(g(y), y, t) \cdot \mathbf{n}(y) = D_2 \nabla u_2(g(y), y, t) \cdot \mathbf{n}(y),$$

plus initial/boundary conditions at $t = 0$, $x = 0, L$ and $y = 0, H$.

Diffusion across an irregular interface

Perturbation solution



Dylan Oliver

Honours student (2021), PhD student (2022–).

QUT Vacation Research Experience Scheme (VRES) Project (2020–2021 Summer).

In collaboration with Mat Simpson (QUT).

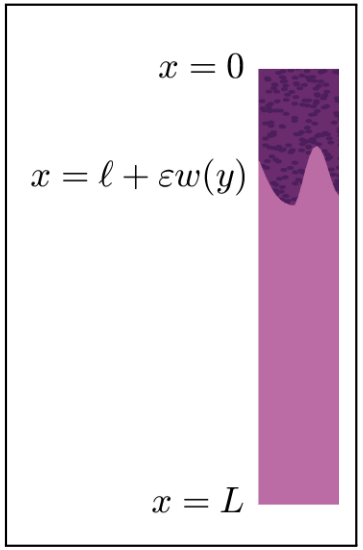
Project Objective: Develop an (approximate) analytical solution when the irregular interface is given by a small perturbation of a (perfectly) horizontal interface, $g(y) = \ell + \varepsilon w(y)$.

$$\text{Top layer:} \qquad u_1(x, y, t) = \sum_{i=0}^{\infty} \varepsilon^i u_1^{(i)}(x, y, t),$$

$$\text{Bottom layer:} \qquad u_2(x, y, t) = \sum_{i=0}^{\infty} \varepsilon^i u_2^{(i)}(x, y, t).$$

Diffusion across an irregular interface

First interface condition



- Continuity of temperature:

$$u_1(\ell + \varepsilon w(y), y, t) = u_2(\ell + \varepsilon w(y), y, t).$$

- Expand in Taylor series:

$$\sum_{k=0}^{\infty} \frac{\varepsilon^k w(y)^k}{k!} \frac{\partial^k u_1}{\partial x^k}(\ell, y, t) = \sum_{k=0}^{\infty} \frac{\varepsilon^k w(y)^k}{k!} \frac{\partial^k u_2}{\partial x^k}(\ell, y, t).$$

- Insert asymptotic expansions:

$$\sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{\varepsilon^{i+k} w(y)^k}{k!} \frac{\partial^k u_1^{(i)}}{\partial x^k}(\ell, y, t) = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{\varepsilon^{i+k} w(y)^k}{k!} \frac{\partial^k u_2^{(i)}}{\partial x^k}(\ell, y, t).$$

- Matching $\mathcal{O}(\varepsilon^n)$ term (when $i + k = n$ or $i = n - k$ for $k = 0, \ldots, n$):

$$u_1^{(n)}(\ell, y, t) = u_2^{(n)}(\ell, y, t) + f^{(n)}(y, t),$$

where $f^{(n)}(y, t)$ depends on (derivatives of) $u_1^{(0)}, u_2^{(0)}, \ldots, u_1^{(n-1)}, u_2^{(n-1)}$.

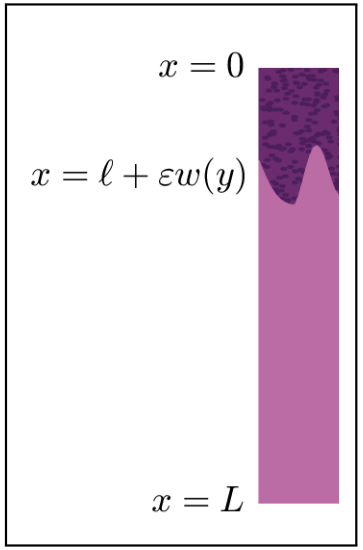
Diffusion across an irregular interface

Second interface condition

- Continuity of diffusive flux:

$$D_1 \nabla u_1(\ell + \varepsilon w(y), y, t) \cdot \mathbf{n}(y) = D_2 \nabla u_2(\ell + \varepsilon w(y), y, t) \cdot \mathbf{n}(y),$$

where normal vector depends on ε with $\boldsymbol{n}(y) = (1, -\varepsilon w'(y))$.



- Expand in Taylor series and insert asymptotic expansions, e.g.,

$$\frac{\partial u_1}{\partial x}(\ell + \varepsilon w(y), y, t) = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{\varepsilon^{i+k} w(y)^k}{k!} \frac{\partial^{k+1} u_1^{(i)}}{\partial x^{k+1}}(\ell, y, t),$$

$$\frac{\partial u_1}{\partial y}(\ell + \varepsilon w(y), y, t) = \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{\varepsilon^{i+k} w(y)^k}{k!} \frac{\partial^{k+1} u_1^{(i)}}{\partial x^k \partial y}(\ell, y, t).$$

- Matching $\mathcal{O}(\varepsilon^n)$ term:

$$D_1 \frac{\partial u_1^{(n)}}{\partial x}(\ell, y, t) = D_2 \frac{\partial u_2^{(n)}}{\partial x}(\ell, y, t) + g^{(n)}(y, t),$$

where $g^{(n)}(y, t)$ depends on (derivatives of) $u_1^{(0)}, u_2^{(0)}, \ldots, u_1^{(n-1)}, u_2^{(n-1)}$.

Diffusion across an irregular interface

Summary

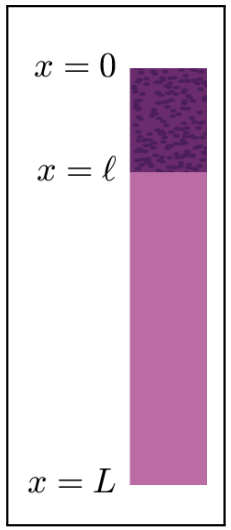
- Perturbation solution (taking a finite number of terms, N):

$$u_1(x,y,t) = \sum_{n=0}^{N-1} \varepsilon^n u_1^{(n)}(x,y,t), \qquad u_2(x,y,t) = \sum_{n=0}^{N-1} \varepsilon^n u_2^{(n)}(x,y,t).$$

- Each term satisfies an initial-boundary value problem on the unperturbed domain:

$$\frac{\partial u_1^{(n)}}{\partial t} = D_1 \Delta u_1^{(n)}, \quad 0 < x < \ell,$$
$$\frac{\partial u_2^{(n)}}{\partial t} = D_2 \Delta u_2^{(n)}, \quad \ell < x < L,$$
$$u_1^{(n)}(\ell,y,t) = u_2^{(n)}(\ell,y,t) + f^{(n)}(y,t),$$
$$D_1 \frac{\partial u_1^{(n)}}{\partial x}(\ell,y,t) = D_2 \frac{\partial u_2^{(n)}}{\partial x}(\ell,y,t) + g^{(n)}(y,t),$$

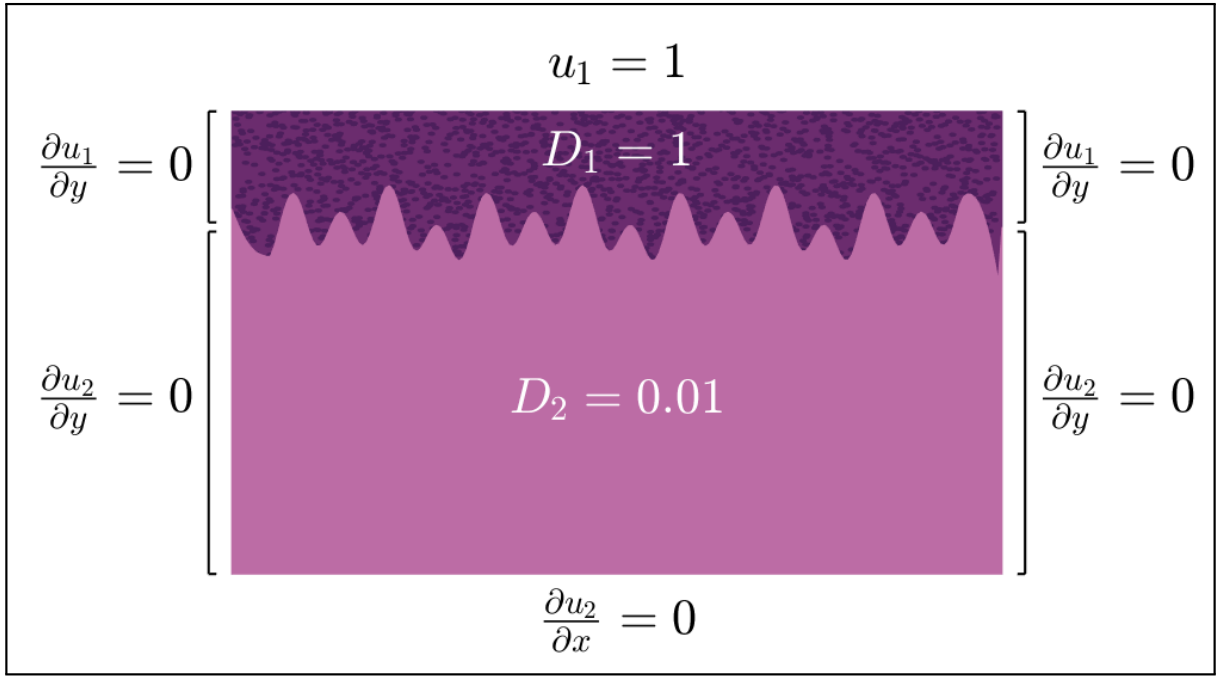
plus initial/boundary conditions at $t = 0$, $x = 0, L$ and $y = 0, H$.



- Initial-boundary value problems must be solved sequentially for $n = 0, 1, \ldots, N-1$.

Diffusion across an irregular interface

Test problem



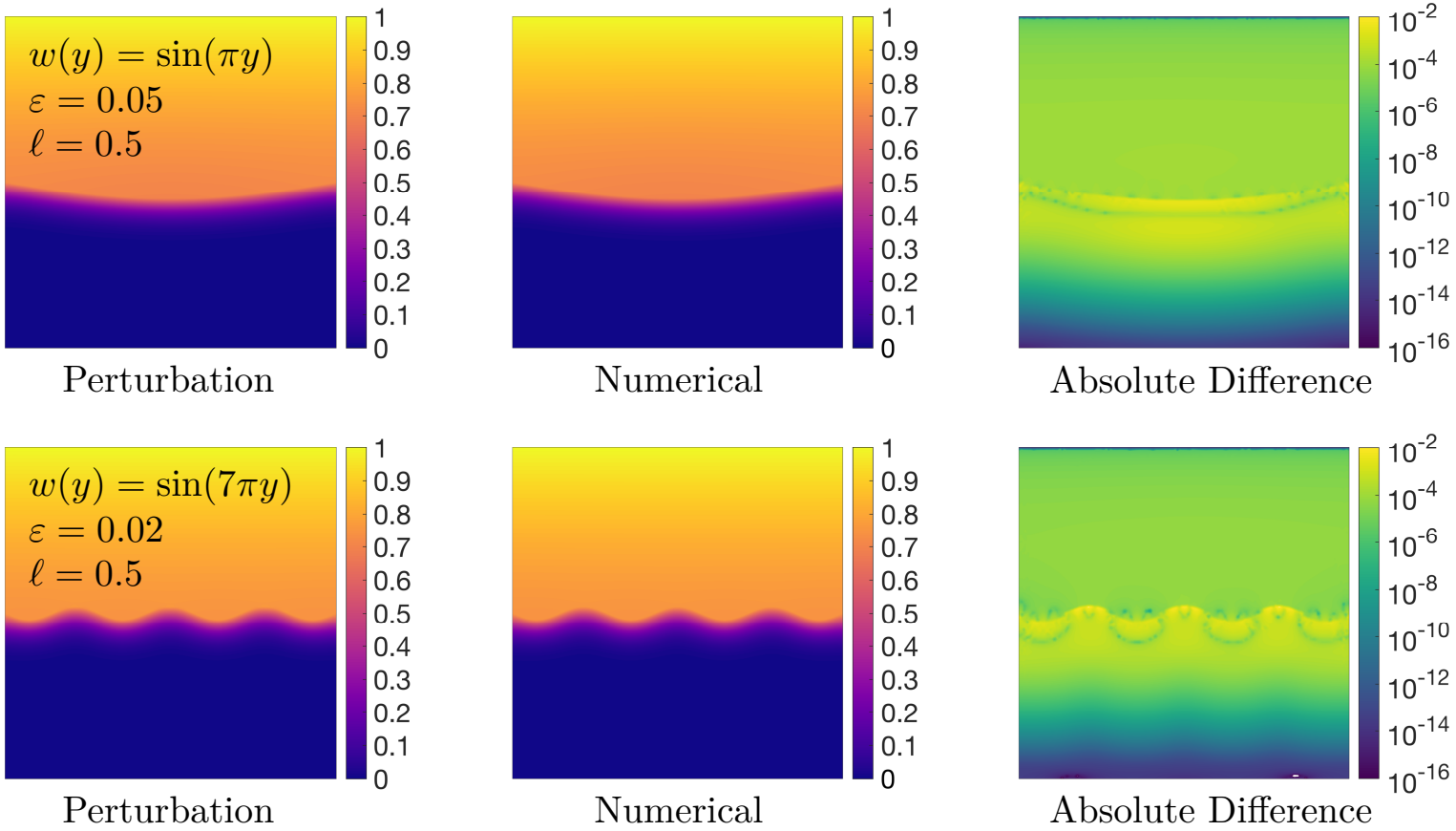
Perturbation solution

- Leading order terms, $u_1^{(0)}$ and $u_2^{(0)}$, satisfy a 1D problem $\rightarrow$ solved using the Laplace transform.
- Higher order terms, $u_1^{(n)}$ and $u_2^{(n)}$ for $n = 1, 2, \ldots, N-1$, satisfy 2D problems $\rightarrow$ solved using Laplace transform and eigenfunction expansion.

Diffusion across an irregular interface

Test problem

Interface at $x = \ell + \varepsilon w(y)$.



Solution comparison at $t = 0.2$.

Diffusion across an irregular interface
Concluding remarks

Summary of research:

- Perturbation solution:

$$u_1(x,y,t) = \sum_{n=0}^{N-1} \varepsilon^n u_1^{(n)}(x,y,t), \qquad u_2(x,y,t) = \sum_{n=0}^{N-1} \varepsilon^n u_2^{(n)}(x,y,t).$$

- If N is held fixed, accuracy improves for decreasing values of ε.
- If ε is held fixed, accuracy **<u>does not</u>** necessarily improve for increasing values of N.

Avenues for future research:

- More general boundary conditions at $x = 0, L$ and $y = 0, H$.
- More general interface conditions at $x = \ell + \varepsilon w(y)$, e.g., imperfect thermal contact.
- Irregular boundary (instead of irregular interface).
- Multiple layers. Multiple irregular interfaces. Multiple irregular interfaces/boundaries.

Diffusion across an irregular interface

Publication, preprint and code

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Research paper

Approximate analytical solution for transient heat and mass transfer across an irregular interface

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ABSTRACT

Motivated by practical applications in heat conduction and contaminant transport, we consider heat and mass diffusion across a perturbed interface separating two finite regions of distinct diffusivity. Under the assumption of continuity of the solution and diffusive flux at the interface, we use perturbation theory to develop an asymptotic expansion of the solution valid for small perturbations. Each term in the asymptotic expansion satisfies an initial–boundary value problem on the unperturbed domain subject to interface conditions depending on the previously determined terms in the asymptotic expansion. Demonstration of the perturbation solution is carried out for a specific, practically-relevant set of initial and boundary conditions with semi-analytical solutions of the initial–boundary value problems developed using standard Laplace transform and eigenfunction expansion techniques. Results for several choices of the perturbed interface confirm the perturbation solution is in good agreement with a standard numerical solution.

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Carr, Oliver and Simpson (2022): doi:10.1016/j.cnsns.2021.106211.
arXiv preprint: arxiv.org/abs/2105.10116.
MATLAB Code: github.com/elliotcarr/Carr2022a.

Diffusion across an irregular interface

Exit time problems on irregular composite domains

Stochastic

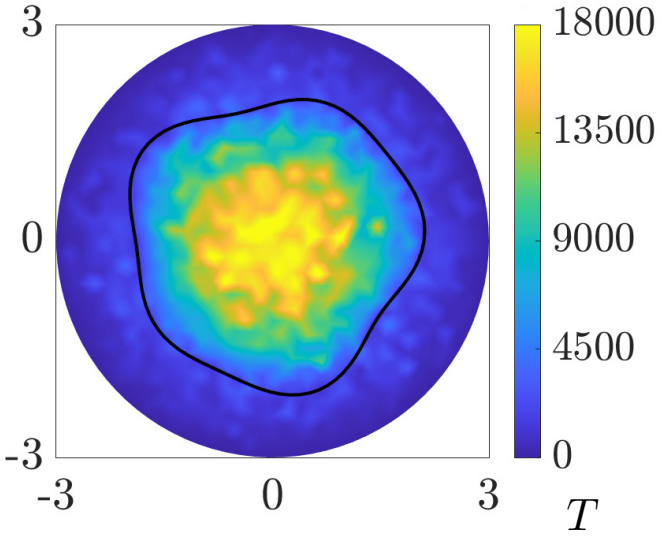
$$T = \frac{1}{N}\sum_{k=1}^{N} t_k,$$

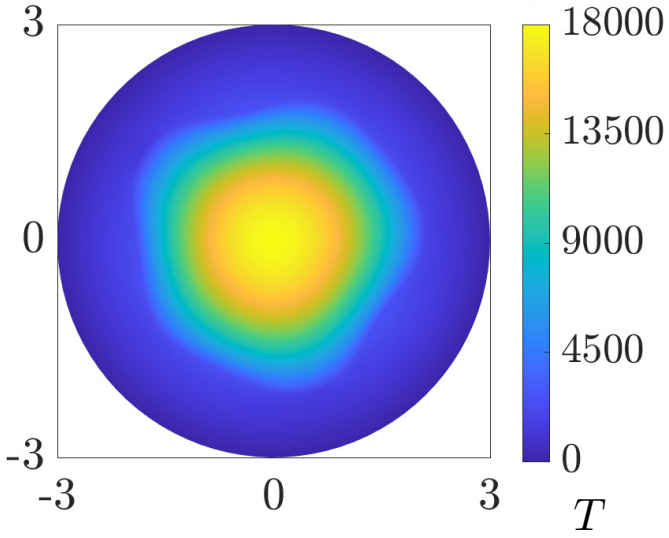
$$P = \begin{cases} P_1, & 0 < r < \mathcal{R}_1(\theta), \\ P_2, & \mathcal{R}_1(\theta) < r < R_2. \end{cases}$$

Deterministic

$$\nabla \cdot [D\nabla T] = -1.$$

$$D = \begin{cases} D_1, & 0 < r < \mathcal{R}_1(\theta), \\ D_2, & \mathcal{R}_1(\theta) < r < R_2. \end{cases}$$





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McInerney, S., Carr, E. J., and Simpson, M. J. (2019). Parameterising continuum models of heat transfer in heterogeneous living skin using experimental data. *International Journal of Heat and Mass Transfer*, 128:964–975.

Simpson, M. J., McInerney, S., Carr, E. J., and Cuttle, L. (2017). Quantifying the efficacy of first aid treatments for burn injuries using mathematical modelling and in vivo porcine experiments. *Scientific Reports*, 7:10925.